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RESEARCH ON STRUCTURAL DYNAMIC TESTING BY IMPEDANCE METHODS. VOLUME II. STRUCTURAL SYSTEM IDENTIFICATION FROM SINGLE-POINT EXCITATION

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Kaman Aerospace Corporation

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RESEARCH ON STRUCTURAL DYNAMIC TESTING BY IMPEDANCE METHODS

VOLUME II
STRUCTURAL SYSTEM IDENTIFICATION FROM SINGLE-POINT EXCITATION

By

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Alex Berman
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November 1972

EUSTIS DIRECTORATE
U. S. ARMY AIR MOBILITY RESEARCH AND DEVELOPMENT LABORATORY
FORT EUSTIS, VIRGINIA

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KAMAN AEROSPACE CORPORATION
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This report contains the theoretical derivation and the presentation of a methodology for system identification of structures. Computer experiments were run to verify this methodology.

The report has been reviewed by this Directorate and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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13. ABSTRACT

The parameters in Lagrange's equations of motion, mass, stiffness, and damping for a mathematical model having fewer degrees of freedom than the linear elastic structure it represents may be determined directly from measured mobility data obtained by forcing the structure at a single point. In conjunction with the mobility data, it is also necessary that the approximate system natural frequencies be known. Thus, using only a minimum amount of impedance-type test data without the use of an intuitive mathematical model, the equations of motion for the complete structure may be obtained. Further, the eigenvector or mode shape, generalized mass, stiffness, and damping associated with each natural frequency are also determined.

A digital computer program was generated to numerically test the aforementioned theory. Computer experiments were conducted to test the sensitivity of the theory to errors in the simulated test data and to determine the practicality of the theory.

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simulated test data						
computer programs						
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Volume II
Structural System Identification From Single-Point Excitation

Final Report

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for

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FORT EUSTIS, VIRGINIA

FOREWORD

The work presented in this report was performed by Kaman Aerospace Corporation under Contract DAAJ02-70-C-0012 (Task 1F162204AA4301) for the Eustis Directorate, U. S. Army Air Mobility Research and Development Laboratory, Fort Eustis, Virginia. The program was implemented under the technical direction of Mr. Joseph H. McGarvey of the Reliability and Maintainability Division* and Mr. Arthur J. Gustafson of the Structures Division.** The report is presented in four volumes, each describing a separate phase of the basic theory of structural dynamic testing using impedance techniques.

Volume I presents the results of an analytical and numerical investigation of the practicality of system identification using fewer measurement points than there are degrees of freedom. The parameters in Lagrange's equations of motion, mass, stiffness, and damping for a mathematical model having fewer degrees of freedom than the linear elastic structure it represents may be determined directly from measured mobility data. Volume II describes the method of system identification wherein the necessary impedance data are experimentally determined by applying a force excitation at a single point on the structure. Volume III presents a method of determining the free-body dynamic responses from data obtained on a constrained structure. Volume IV describes a method of obtaining the equations for the combination of measured mobility matrices of a helicopter and its subsystems. The response of the combination of a helicopter and its subsystems is determined from data based on the experimental results of the main system and subsystems separately.

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*Division name changed to Military Operations Technology Division.

**Division name changed to Technology Applications Division.

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LIST OF SYMBOLS

C	influence coefficient
d	damping
f	force
$\tilde{f}$	force phasor
g	structural damping coefficient
i	imaginary operator ($i = \sqrt{-1}$)
K	stiffness
$\mathcal{K}$	modal stiffness, generalized stiffness
m	mass
$\mathcal{M}$	modal mass, generalized mass
R	residual, defined in text
S	modal mobility ratio, defined in text
Y	displacement mobility, $\partial\tilde{y}/\partial\tilde{f}$
[Φ]	matrix of modal vectors

BRACKETS

[ ], ( )	matrix
⌊ ⌋	diagonal matrix
{ }	column or row vector

SUPERSCRIPTS

(q)	q-th iteration
*	modal parameter
R	real

LIST OF SYMBOLS (Continued)

I	imaginary
T	transpose
-1	inverse
-T	transpose of the inverse
+	pseudoinverse, generalized inverse, generalized reciprocal

SUBSCRIPTS

( )	a subscripted index in parentheses means the index is held constant
i	modal index
j	degree of freedom index, generalized coordinate index
k	degree of freedom index, generalized coordinate index

OTHER INDICES

N	number of degrees of freedom
Q	number of modes
P	number of forcing frequencies
J	number of generalized coordinates
JxP	capital letters under matrices indicate the number of rows and columns respectively
.	a dot over a quantity indicates differentiation with respect to time

INTRODUCTION

The success of a helicopter structural design is highly dependent on the ability to predict and control the dynamic response of the fuselage and mechanical components. Conventionally, this involves the formulation of intuitively based equations of motion. Ideally, this process would reduce the physical structure to an analytical mathematical model which would predict accurately the dynamic response characteristics of the actual structure. Obviously, the creation of such an intuitive abstraction of a complicated real structure requires considerable expertise and inherently includes a high degree of uncertainty. Structural dynamic testing is required to substantiate the analytical results. The analysis is modified until successful correlation is obtained between the analytical predictions and the test results.

This report describes the theory of structural dynamic testing using impedance techniques as applied to a mathematical model having fewer degrees of freedom than the structure it represents. The test information is obtained with single point excitation of the model. Reference 1 describes the method of obtaining a model directly from test measurements for a hypothetical structure which has the same number of degrees of freedom as the mathematical model. In reality, the number of degrees of freedom of a physical structure is infinite, therefore, the usefulness of model identification, necessarily with a finite number of degrees of freedom, using impedance testing techniques depends on the ability to simulate the real structure with a small mathematical model. Reference 2 illustrates the method of obtaining a model, using impedance testing techniques, that is comprised of less degrees of freedom than the physical structure it approximates. That method required measured mobility data obtained at selected points of the structure with the force input applied at each of the prescribed locations. The present theory is similar to that of Reference 2 except that the excitation is applied at only one point on the model, thereby substantially reducing the mobility data essential to the analysis.

The process of deriving the equations of motion from test data is referred to as system identification. The only input information required in this theory is measured mobility data obtained with the excitation at only one point on the model and the approximate natural frequency of each

mode. This information can be readily obtained from impedance testing of the actual structure over the frequency range of interest yielding the second order, structurally damped linear equations of motion.

System identification theories of any practical engineering significance must be functional with a reasonable degree of experimental error. In this report, a series of computer experiments incorporating experimental errors was documented. This report presents an extension of the analysis derived in Reference 2 whereby an identified model with a finite number of degrees of freedom, obtained from impedance type testing with excitation at only one point on the structure, simulates the actual structure wherein the number of degrees of freedom is infinite.

THEORY

DERIVATION OF THE SINGLE-POINT ITERATION PROCESS

As indicated in References 1 and 2, the mobility of a structure is given by

$$[Y_{\omega}] = [\Phi]\lceil Y^{*}_{i(\omega)}\rfloor[\Phi]^{T} \tag{1}$$

With excitation at station k, the responses at station j, including k, are obtained. These provide the k-th column of the mobility at a particular forcing frequency ω_1:

$$\{Y_{j(k)1}\} = \sum_{i=1}^{N} Y^{*}_{i(1)}\phi_{ki}\{\phi\}_i = [\Phi]\{Y^{*}_{i1}\phi_{ki}\} \tag{2}$$

$$1 \leq j \leq J, \; 1 \leq i \leq N$$

This represents a column of mobility values, each element of which is the response at a point of interest on the structure with excitation at station k and at forcing frequency ω_1.

Similarly, with the exciter remaining at station k, the k-th column of the mobility at another frequency, ω_2, can be obtained:

$$\{Y_{j(k)2}\} = \sum_{i=1}^{N} Y^{*}_{i(2)}\phi_{ki}\{\phi\}_i = [\Phi]\{Y^{*}_{i2}\phi_{ki}\} \tag{3}$$

The mobility columns represented by (2) and (3) may be combined into one matrix:

$$\underset{Jx2}{[\{Y_{j(k)1}\}\{Y_{j(k)2}\}]} = [\Phi][\{Y^{*}_{i1}\phi_{ki}\}\{Y^{*}_{i2}\phi_{ki}\}]$$

$$= \underset{JxN}{[\Phi]}\;\underset{NxN}{\lceil\phi_{ki}\rfloor}\;\underset{Nx2}{[\{Y^{*}_{i1}\}\{Y^{*}_{i2}\}]} \tag{4}$$

In general, for P forcing frequencies ($1 \leq p \leq P$),

$$\underset{JxP}{[Y_{j(k)p}]} = \underset{JxN}{[\Phi]}\;\underset{NxN}{\lceil\phi_{ki}\rfloor}\;\underset{NxP}{[Y^{*}_{ip}]} \tag{5}$$

If J > P, Equation (5) is a set of more equations than unknowns for which there is no solution. Equation (5) can then be written as

$$\underset{J\times P}{[Y_{j(k)p}]} = \underset{J\times N}{[\Phi]} \underset{N\times N}{\lceil \phi_{ki} \rfloor} \underset{N\times P}{[Y^{*}_{ip}]} + \underset{J\times p}{[R_{jp}]} \tag{6}$$

where R_{jp} is the residual associated with the j-th station and the p-th forcing frequency.

As described in References 1 and 2, the imaginary displacement mobility contains significant information relating to modes associated with natural frequencies in proximity to the forcing frequency. As shown in Reference 3, accurate estimates of the modal vectors may be obtained by considering only the effects of modes proximate to the forcing frequency. Therefore the analysis will employ only Q modes, where Q is less than N. Consider the imaginary displacement mobility

$$[Y^{I}_{j(k)p}] = [\Phi] \lceil \phi_{ki} \rfloor [Y^{*I}_{ip}] + [R_{jp}] \tag{7}$$

The dominant element in each row of the $[Y^{*I}_{ip}]$ matrix will be the modal mobility measured at the forcing frequency in proximity to a particular natural frequency. Normalizing the rows of the aforementioned matrix on the largest element yields

$$[S_{ip}] = [\frac{Y^{*I}_{ip}}{Y^{*I}_{in}}] \tag{8}$$

where Y^{*I}_{in} is the maximum value of the i-th row. Equation (7) may be rewritten, incorporating Equation (8):

$$[Y^{I}_{j(k)p}] = [\Phi] \lceil \phi_{ki} Y^{*I}_{in} \rfloor [S_{ip}] + [R_{jp}] \tag{9}$$

The $[S_{ip}]$ matrix can be evaluated by considering the expression for the imaginary displacement modal mobility

$$Y^{*I}_{i(\omega)} = - \frac{g_i}{m_i \Omega_i^2 \{g_i^2 + (1 - \frac{\omega^2}{\Omega_i^2})^2\}} \tag{10}$$

Therefore from Equation (8),

$$S_{ip} = \frac{g_i^2 + (1 - \frac{\omega_n^2}{\Omega_i^2})^2}{g_i^2 + (1 - \frac{\omega_p^2}{\Omega_i^2})^2} \tag{11}$$

Because g_i, the structural damping coefficient of the i-th mode, is generally quite small, typically of the order 5 percent, the [S] matrix can be accurately estimated by assuming $g_i = 0$, thus, requiring knowledge of only the forcing frequencies and the natural frequencies. It will be shown that an accurate estimate of S is not necessary, although helpful, as iterations will converge on the best values in S in the least-squares sense.

The matrix Equation (9) has no solution. An approximation to a solution may be defined as that which makes the Euclidian norm of the matrix of residuals a minimum. This, as will be proved later, is given through use of the pseudo-inverse.

Equation (9) will be solved utilizing matrix iteration techniques using $[S^{(0)}_{ip}]$ as a first estimate. As indicated in the following sections, the modal vector matrix with respect to which the Euclidian norm of the residuals is a minimum is given by

$$[\Phi^{(1)}] = [Y^I_{j(k)p}][S^{(0)}_{ip}]^+ [\frac{1}{\phi_{ki} Y^{*I}_{in}}] \tag{12}$$

where $[S^{(0)}_{ip}]^+$ is defined as the generalized inverse or pseudoinverse of $[S^{(0)}]$ and is given by

$$[S_{ip}^{(0)}]^{+} = [S_{ip}^{(0)}]^{T}([S_{ip}^{(0)}][S_{ip}^{(0)}]^{T})^{-1} \tag{13}$$

where

$$[S_{ip}^{(0)}][S_{ip}^{(0)}]^{+} = [I_L]$$

It follows then that

$$[Y_{j(k)p}^{I}] = [\Phi^{(1)}]\lceil\phi_{ki}Y_{in}^{*I}\rfloor[S_{ip}^{(0)}] + [R_{jp}^{(0)}] \tag{14}$$

in which the Euclidian norm of $[R_{jp}^{(0)}]$ is a minimum with respect to $[\Phi^{(1)}]$.

Using $[\Phi^{(1)}]$, a matrix $[S_{ip}^{(1)}]$ can be found to give an equation

$$[Y_{j(k)p}^{I}] = [\Phi^{(1)}]\lceil\phi_{ki}Y_{in}^{*I}\rfloor[S_{ip}^{(1)}] + [R_{jp}^{(1)}] \tag{15}$$

such that the Euclidian norm of $[R_{jp}^{(1)}]$ is a minimum with respect to $[S_{ip}^{(1)}]$. This is given by

$$[S_{ip}^{(1)}] = \left\lceil\frac{1}{\phi_{ki}Y_{in}^{*I}}\right\rfloor[\Phi^{(1)}]^{+}\ [Y_{j(k)p}^{I}] \tag{16}$$

where

$$[\Phi]^{+} = ([\phi]^{T}[\phi])^{-1}[\phi]^{T} \text{ and } [\phi]^{+}[\phi] = [I_R] \tag{17}$$

It is apparent from the first cycle of the iteration, by comparing Equations (11) and (15), that the process consists of alternately dealing with the left and right identity matrices. At each successive iteration, a solution is found that minimizes the Euclidian norm of the residual matrix with respect to the newly found matrix of either [S] or [Φ].

In simplified notation, the q-th iteration becomes

$$[\phi^{(q)}] = [Y^I][S^{(q-1)}]^+ [\frac{1}{\phi_{ki} Y_{in}^{*I}}] \tag{18}$$

and

$$[S^{(q)}] = [\frac{1}{\phi_{ki} Y_{in}^{*I}}][\phi^{(q)}]^+[Y^I]$$

The next iteration is

$$[\phi^{(q+1)}] = Y^I[S^{(q)}]^+ [\frac{1}{\phi_{ki} Y_{in}^{*I}}]$$

$$[S^{(q+1)}] = [\frac{1}{\phi_{ki} Y_{in}^{*I}}][\phi^{(q+1)}]^+[Y^I] \tag{19}$$

This is the basic algorithm used in the matrix iteration procedure.

DETERMINING THE MODAL PARAMETERS

From Equation (6) of the previous section, one column, which is at a particular forcing frequency, p, with the excitation at station k, can be written as

$$\{Y_{j(kp)}\} = [\Phi]\lceil\phi_{ki}\rfloor\{Y^{*I}_{i(p)}\} + \{R_{j(p)}\} \tag{20}$$

The number of modes, Q, included in Equation (20) cannot be greater than the number of points of interest on the specimen, J, and generally will be much less since only those modes which have significant effect on the mobility at the forcing frequency, ω_p, will be considered. Ordinarily, the number of modes used will not be greater than 3 or 4 for any given forcing frequency, and these will be the modes in the vicinity of the forcing frequency in question.

The real and imaginary modal mobilities are calculated from

$$\{Y^{*R}_{i(p)}\} = \lceil\frac{1}{\phi_{ki}}\rfloor[\Phi]^{+}\{Y^{R}_{j(kp)}\} \tag{21}$$

and

$$\{Y^{*I}_{i(p)}\} = \lceil\frac{1}{\phi_{ki}}\rfloor[\Phi]^{+}\{Y^{I}_{j(kp)}\} \tag{22}$$

From Reference 1 the real displacement mobility can be calculated as

$$Y^{*R}_{i\omega_p} = \frac{1}{K_i}\,\frac{1-\omega_p^{\,2}/\Omega_i^{\,2}}{g_i^{\,2} + (1-\omega_p^{\,2}/\Omega_i^{\,2})^2} \tag{23}$$

and the imaginary modal mobility by

$$Y^{*I}_{i\omega_p} = \frac{1}{K_i}\,\frac{-g_i}{g_i^{\,2} + (1-\omega_p^{\,2}/\Omega_i^{\,2})^2} \tag{24}$$

The real modal impedance can be written as

$$Z^{*R}_{i\omega_p} = \frac{Y^{*R}_{i\omega_p}}{(Y^{*R}_{i\omega_p})^2 + (Y^{*I}_{i\omega_p})^2} \qquad (25)$$

Substituting Equations (23) and (24) into (25) yields

$$Z^{*R}_{i\omega_p} = \mathcal{K}_i (1 - \omega_p{}^2/\Omega_i{}^2) \qquad (26)$$

From Equation (26) it is observed that the modal impedance is a linear function of the square of the forcing frequency.

The forcing frequency at which the modal impedance becomes zero is, therefore, the natural frequency. From a least-squares analysis of modal impedance as a function of forcing frequency squared, proximate to the natural frequency, the generalized stiffness of the i-th mode and the natural frequency of the i-th mode can be calculated.

The generalized mass associated with the i-th mode is given by

$$\mathcal{M}_i = \mathcal{K}_i/\Omega_i{}^2 \qquad (27)$$

The structural damping coefficient may be determined from

$$g_i = \left(\frac{\omega_p{}^2}{\Omega_i{}^2} - 1\right) \frac{Y^{*I}_{i\omega_p}}{Y^{*R}_{i\omega_p}} \qquad (28)$$

EQUATIONS OF MOTION

There are two basic types of dynamic mathematical models describing structures. The conventional type, covering as many modes as degrees of freedom, is called "Complete Models" and is considered in References 1 and 2. The other type labelled "Incomplete Models" considers fewer modes than points of interest on the structure and was first described in Reference 5. Using the methods described herein, it is possible to identify either a complete model or a form of incomplete model.

Incomplete Models

Consider a rectangular identified modal matrix which has J rows indicating the points of interest on the structure and Q columns representing the modes being considered where J > Q. The influence coefficient matrix for the incomplete model is given by

$$[C_{inc}] = [\Phi]\left\lceil\frac{1}{K_i}\right\rfloor[\Phi]^T \tag{29}$$

The above matrix, similar to all incomplete model parameter matrices, is singular, being of rank Q and order J. The mass, stiffness and damping matrices for the incomplete model are

$$\begin{aligned}
[m_{inc}] &= [\Phi]^{+T}\lceil M_i \rfloor[\Phi]^+ \\
[K_{inc}] &= [\Phi]^{+T}\lceil K_i \rfloor[\Phi]^+ \qquad (30)\\
[d_{inc}] &= [\Phi]^{+T}\lceil g_i K_i \rfloor[\Phi]^+
\end{aligned}$$

The classical modal eigenvalue equation has the analogous incomplete form

$$[c_{inc}][m_{inc}]\{\phi_i\} = \frac{1}{\Omega_i^2}\{\phi_i\} \tag{31}$$

Complete Models

For the complete model the identified modal vector matrix is square, having the same number of degrees of freedom as mode shapes; that is, J = Q. The influence coefficient matrix is given by

$$[c] = [\Phi][1/\mathcal{K}_i\rfloor[\Phi]^T = \sum_{i=1}^{N} \frac{1}{\mathcal{K}_i}\{\phi_i\}\{\phi_i\}^T \qquad (32)$$

The mass, stiffness and damping matrices for the complete model are

$$[m] = [\phi]^{-T}\lceil m_i\rfloor[\phi]^{-1}$$

$$[k] = [\phi]^{-T}\lceil \mathcal{K}_i\rfloor[\phi]^{-1}$$

$$[d] = [\phi]^{-T}\lceil g_i\,\mathcal{K}_i\rfloor[\phi]^{-1} \qquad (33)$$

as indicated in Reference 1.

Full Mobility Matrix

The full mobility matrix of either complete or incomplete models is given by

$$[Y] = [\Phi]\lceil Y_i^*\rfloor[\Phi]^T \qquad (34)$$

where for the complete model the $[\Phi]$ matrix is square, having J columns and J rows. However, in the case of the incomplete model the modal matrix $[\Phi]$ is rectangular, having J rows and Q columns, where $J > Q$.

PROOF THAT THE PSEUDOINVERSE MINIMIZES THE NORM OF THE RESIDUALS

Take the transpose of Equation (9) and write the equation for one column of the transpose of the mobility matrix:

$$[Y^{I}_{j(k)p}]^{T} = [S_{ip}]^{T}[\phi_{ji}]^{T} + [R_{jp}]^{T}$$

$$\{Y_{(jk)p}\} = [S_{ip}]^{T}\{\phi_{(j)i}\} + \{R_{(j)p}\} \tag{35}$$

$$\{R_{(j)p}\} = \{Y_{(jk)p}\} - [S_{ip}]^{T}\{\phi_{(j)i}\} \tag{36}$$

$$\{R_{(j)p}\}^{T}\{R_{(j)p}\} = \{Y_{(jk)p}\}^{T}\{Y_{(jk)p}\} - \{Y_{(jk)p}\}^{T}[S_{ip}]^{T}\{\phi_{(j)i}\} - \{\phi_{(j)i}\}^{T}[S_{ip}]\{Y_{(jk)p}\} + \{\phi_{(j)i}\}^{T}[S_{ip}][S_{ip}]^{T}\{\phi_{(j)i}\} \tag{37}$$

Equation (37) is, of course, a scalar product and it is recognized that the derivative of a scalar with respect to a vector is a vector; in other words, Equation (36) is a vector in p-dimensional space and Equation (37) is its dot produce on itself - that is, its length squared. We wish to find the vector $\{\phi\}$ which makes the length of the residuals vector a minimum.

Take the partial derivative of Equation (37) with respect to $\{\phi_{(j)i}\}^{T}$ and set equal to zero to obtain the modal vector for which the Euclidian norm of the residuals is a minimum:

$$0 = -2[S^{(0)}_{ip}]\{Y_{(jk)p}\} + 2[S^{(0)}_{ip}][S^{(0)}_{ip}]^{T}\{\phi^{(1)}_{(j)i}\}$$

or

$$\{\phi^{(1)}_{(j)i}\} = ([S^{(0)}_{ip}][S^{(0)}_{ip}]^{T})^{-1}[S^{(0)}_{ip}]\{Y_{(jk)p}\} \tag{38}$$

and

$$\{\phi^{(1)}_{(j)i}\}^{T} = \{Y_{(jk)p}\}^{T}[S^{(0)}_{ip}]^{T}([S^{(0)}_{ip}][S^{(0)}_{ip}]^{T})^{-1} \tag{39}$$

as the inverted matrix is symmetrical. Equation (39) is any row in Equation (12). The sum of the minimum Euclidian norms of the rows of a matrix is, by definition, the minimum Euclidian norm of the matrix, and it therefore follows from Equation (39) that

$$[\Phi^{(1)}] = [Y_{j(k)p}][S_{ip}^{(0)}]^T([S_{ip}^{(0)}][S_{ip}^{(0)}]^T)^{-1}$$

which is given by Equations (12) and (13). Q.E.D. The basic observation which makes the above proof of the pseudoinverse possible should be credited to Klosterman, Reference (4).

To show that the [S] matrix obtained using the pseudoinverse of [Φ] minimizes the norm of the residual, write the equation for a column of Equation (9):

$$\{Y_{j(kp)}\} = [\Phi]\{S_{i(p)}\} + \{R_{j(p)}\}$$

$$\{R_{j(p)}\} = \{Y_{j(kp)}\} - [\Phi]\{S_{i(p)}\} \tag{40}$$

$$\begin{aligned}\{R_{j(p)}\}^T\{R_{j(p)}\} = \{Y_{j(kp)}\}^T\{Y_{j(kp)}\} - \{Y_{j(kp)}\}^T[\Phi]\{S_{i(p)}\} \\ - \{S_{i(p)}\}^T[\Phi]^T\{Y_{j(kp)}\} + \{S_{i(p)}\}^T[\Phi]^T[\Phi]\{S_{i(p)}\}\end{aligned} \tag{41}$$

Set $\dfrac{\partial\{R_{j(p)}\}^T\{R_{j(p)}\}}{\partial\{S_{i(p)}\}^T} = 0$ and solve for $\{S_{i(p)}^{(1)}\}$

$$\{S_{i(p)}^{(1)}\} = ([\Phi]^T[\Phi]^{-1}\ [\Phi]^T\{Y_{j(kp)}\} \tag{42}$$

or

$$[S_{ip}^{(1)}] = ([\Phi]^T[\Phi])^{-1}[\Phi]^T[Y_{j(k)p}] \tag{43}$$

which is the same as Equation (16). Q.E.D.

PROOF THAT ITERATIONS USING THE PSEUDOINVERSE OF S AND Φ CONVERGE MONOTONICALLY ON MINIMUM SUM OF RESIDUAL SQUARES

In the q-th iteration, where q is odd,

$$[Y^{I}_{j(k)p}] = [\phi^{(q-1)}][S^{(q-1)}_{ip}] + [R^{(q-1)}_{jp}] \tag{44}$$

$$[\phi^{(q)}] \equiv [Y^{I}_{j(k)p}][S^{(q-1)}_{ip}]^{+} = [\phi^{(q-1)}] + [R^{(q-1)}_{jp}][S^{(q-1)}]^{+} \tag{45}$$

because $[S][S]^{+} = [I_L]$. Then

$$[Y^{I}_{j(k)p}] = [\phi^{(q)}][S^{(q-1)}_{ip}] + [R^{(q)}_{jp}] \tag{46}$$

Substitute Equation (45) into Equation (46):

$$[Y^{I}_{j(k)p}] = [\phi^{(q-1)}][S^{(q-1)}_{ip}] + [R^{(q-1)}_{jp}][S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}] + [R^{(q)}_{jp}] \tag{47}$$

or

$$[Y^{I}_{j(k)p}] = [Y^{I}_{j(k)p}] - [R^{(q-1)}_{jp}] + [R^{(q-1)}_{jp}][S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}] + [R^{(q)}_{jp}]$$

Therefore

$$\underset{J\times P}{[R^{(q)}_{jp}]} = \underset{J\times P}{[R^{(q-1)}_{jp}]}(\underset{P\times P}{[I]} - [S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}]) \tag{48}$$

The p-th row of $[R^{(q)}_{jp}]$ is

$$\{R^{(q)}_{j(p)}\}^{T} = \{R^{(q-1)}_{j(p)}\}^{T}([I] - [S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}])$$

$$\{R^{(q)}_{j(p)}\}^{T}\{R^{(q)}_{j(p)}\} = \{R^{(q-1)}_{j(p)}\}^{T}([I] - [S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}])([I] - [S^{(q-1)}_{ip}]^{+}[S^{(q-1)}_{ip}])^{T}\{R^{(q-1)}_{j(p)}\}$$

But $[I] - [S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}]$ is symmetrical and, from Equation (13),

$$[S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^+ = [I_L]. \text{ Therefore,}$$

$$\{R_{j(p)}^{(q)}\}^T \{R_{j(p)}^{(q)}\} = \{R_{j(p)}^{(q-1)}\}^T \{R_{j(p)}^{(q-1)}\}$$

$$- \{R_{j(p)}^{(q-1)}\}^T [S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}] \{R_{j(p)}^{(q-1)}\} \qquad (49)$$

$[S_{ip}^{(q-1)}]$ is maximally ranked in its rows, of rank Q where $1 \le i \le Q$. Therefore $[S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T$ and its square root $([S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T)^{1/2}$ are nonsingular of rank Q and symmetrical. Now, $[S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}]$ is real, symmetric and singular. It is known that a real symmetric matrix [A] of rank Q is positive semidefinite if and only if there exists a matrix [C] of rank Q such that $[A] = [C]^T [C]$. Let $([S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T)^{-1/2} [S_{ip}^{(q-1)}] \equiv [C]$, rectangular of rank Q.

$$[S_{ip}^{(q-1)}]^T ([S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T)^{-\frac{T}{2}} ([S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T)^{-\frac{1}{2}} [S_{ip}^{(q-1)}]$$

$$= C^T C = [S_{ip}^{(q-1)}]^T ([S_{ip}^{(q-1)}][S_{ip}^{(q-1)}]^T)^{-1} [S_{ip}^{(q-1)}]$$

$$= [S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}] \qquad (50)$$

Therefore $[S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}]$ is positive semidefinite and

$\{R_j^{(q-1)}\}^T [S_{ip}^{(q-1)}]^+ [S_{ip}^{(q-1)}] \{R_{j(p)}^{(q-1)}\}$ in Equation (49) must be a nonnegative number. But the first term on the right side and the left side of Equation (49) are also necessarily nonnegative. Therefore

$$\{R_{j(p)}^{(q)}\}^T\{R_{j(p)}^{(q)}\} < \{R_{j(p)}^{(q-1)}\}^T\{R_{j(p)}^{(q-1)}\} \text{ and}$$

$$\sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q)})^2 < \sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q-1)})^2 \tag{51}$$

For the alternate calculation, q odd

$$[S_{ip}^{(q)}] = [\phi^{(q)}]^+[Y_{j(k)p}^I] \tag{18}$$

But $[Y_{j(k)p}^I] = [\Phi^{(q)}][S_{ip}^{(q-1)}] + [R_{jp}^{(q)}]$, so (46)

$$[S_{ip}^{(q)}] = [S_{ip}^{(q-1)}] + [\phi^{(q)}]^+[R_{jp}^{(q)}] \tag{52}$$

Substituting $[S_{ip}^{(q)}]$ for $[S_{ip}^{(q-1)}]$, we obtain

$$[Y_{j(k)p}^I] = [\phi^{(q)}][S_{ip}^{(q)}] + [R_{jp}^{(q+1)}]$$

$$= [\phi^{(q)}][S_{ip}^{(q-1)}] + [\phi^{(q)}][\phi^{(q)}]^+[R_{jp}^{(q)}] + [R_{jp}^{(q+1)}] \tag{53}$$

From Equations (46) and (53),

$$[Y_{j(k)p}^I] = [Y_{j(k)p}^I] - [R_{jp}^{(q)}] + [\phi^{(q)}][\phi^{(q)}]^+[R_{jp}^{(q)}] + [R_{jp}^{(q+1)}]$$

or

$$[R_{jp}^{(q+1)}] = ([I] - [\phi^{(q)}][\phi^{(q)}]^+)[R_{jp}^{(q)}] \tag{54}$$

Compare Equation (54) to Equation (48).

Consider a column of Equation (54) $\{R_{j(p)}^{(q+1)}\}$. Because of Equation (18),

$$\frac{\partial\{R_{j(p)}^{(q+1)}\}^T\{R_{j(p)}^{(q+1)}\}}{\partial\{S_{i(p)}\}} = 0$$

$$\{R_{j(p)}^{(q+1)}\}^T\{R_{j(p)}^{(q+1)}\} = \{R_{j(p)}^{(q)}\}^T([I] - [\Phi^{(q)}][\Phi^{(q)}]^+)^T([I]$$

$$- [\Phi^{(q)}][\Phi^{(q)}]^+)\{R_{j(p)}^{(q)}\} = \{R_{j(p)}^{(q)}\}^T\{R_{j(p)}^{(q)}\}$$

$$- \{R_{j(p)}^{(q)}\}[\Phi^{(q)}][\Phi^{(q)}]^+\{R_{j(p)}^{(q)}\} \qquad (55)$$

because $[\Phi]^+[\Phi] = [I_R]$ (Equation 15) and $[\Phi^{(q)}][\Phi^{(q)}]^+$ is symmetrical. Now $[\Phi^{(q)}][\Phi^{(q)}]^+ = [\Phi^{(q)}]([\Phi^{(q)}])^{-\frac{T}{2}}$ $([\Phi^{(q)}])^{-1/2}[\Phi^{(q)}]^T$ and $[\Phi^{(q)}]$ is necessarily maximally column ranked. Therefore, $[\Phi^{(q)}][\Phi^{(q)}]^+$ is positive semi-definite. The left side of Equation (55) is the positive difference between two positive numbers, and it follows that

$$\sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q+1)})^2 < \sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q)})^2 \qquad (56)$$

Equation (51) shows that the Euclidian norm of residuals with odd index q is less than the norm of residuals of index q-1; Equation (56) shows that the norm of residuals of index q+1 is less than the norm of residuals of index q. Equations (51) and (56) show that it is immaterial whether q is odd or even.

$$\sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q+1)})^2 < \sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q)})^2 < \sum_{j=1}^{J}\sum_{p=1}^{P}(R_{jp}^{(q-1)})^2 \qquad (57)$$

Equation (57) covers a complete iteration cycle. Q.E.D.

NOTE ON THE DERIVATIVE OF A SCALAR WITH RESPECT TO A VECTOR

Let [S] be a square matrix of order R

$$\{\chi\}^T[S]\{y\} = \sum_{i=1}^{R}\sum_{j=1}^{R} S_{ij}\chi_i y_j$$

$$\{y\}^T[S]^T\{\chi\} = \sum_{i=1}^{R}\sum_{j=1}^{R} S_{ji}y_i\chi_j$$

$$\frac{\partial\{\chi\}^T[S]\{y\}}{\partial\{\chi\}^T} = \sum_{j=1}^{R} S_{ij}y_j = [S]\{y\}$$

$$-\frac{\partial\{\chi\}^T[S]\{y\}}{\partial\{y\}} = \sum_{i=1}^{R} S_{ij}\chi_i = [S]^T\{\chi\}$$

$$\frac{\partial\{\chi\}^T[S]\{y\}}{\partial\{y\}^T} = \frac{\partial\{y\}^T[S]^T\{\chi\}}{\partial\{y\}^T} = \frac{\partial \sum_{i=1}^{R}\sum_{j=1}^{R} S_{ji}y_i\chi_j}{\partial\{y\}^T} = [S]^T\{\chi\}$$

$$\frac{\partial\{\chi\}^T[S]\{\chi\}}{\partial\{\chi\}^T} = [S]\{\chi\} + [S]^T\{\chi\} = ([S] + [S]^T)\{\chi\}$$

IDENTIFIED GENERALIZED MASSES

Typical generalized mass identifications are shown in Tables I through VI. Table VII describes the various models for which data is presented in Tables I through VI. Table VIII presents a lumped mass description of the twenty-point specimen which was used to generate the simulated experimental data. The model stations used in the various models refer to the corresponding stations in the twenty-point specimen. Table I presents results for model 5C, which are typical of the results obtained for other five-point models. Data are presented for conditions of zero experimental error and for simulated experimental displacement mobility data recorded with a random error of ± 5 percent and a bias error of +5 percent. For the cases involving error, the random displacement error was computed using a uniformly distributed probability density function. This error was applied to both the real and imaginary components of the displacement mobility data. Table I presents the effects of random number, the seed used in generating the random error. The results indicate the method is extremely insensitive to measurement errors as applied herein.

Table II shows results for several different five-point models. It is apparent that no outstanding differences exist among the models considered. The results for the twenty-point specimen, the simulated actual structure, are also given in the table for comparison. The generalized mass distribution associated with each of the models is in excellent agreement with the twenty-point results.

Tables III and IV present results for the nine-point models studied. Again, the calculations of the generalized masses for the various nine-point models under consideration are in agreement with the simulated structure.

Tables V and VI describe the results of the computer experiments conducted employing the twelve-point models. The calculations produced acceptable results except for identification of the generalized masses of the 10th and 11th modes. The generalized masses associated with these models are extremely small in comparison to the remaining modal generalized masses. Further, the mode shape of the 10th mode indicates lack of response at all points of interest on the structure other than the first station. Therefore, the effect of the 10th mode is difficult to evaluate in the calculation of the generalized parameters.

TABLE I. IDENTIFICATION OF GENERALIZED MASSES, 5 X 5 MODEL* OF 20 X 20 SPECIMEN

Computer Experiment Number		290	291	292	293	294	1**
Random Disp. Error		0	$\pm5\%$	$\pm5\%$	$\pm5\%$	$\pm5\%$	0
Bias Disp. Error		0	+5%	+5%	+5%	+5%	0
Random Error Seed		-	5	13	421	1094	-
Stations (In.)	Mode	Generalized Masses ($Lb\text{-}Sec^2/In.$)					
0	1	8.415	8.560	8.543	8.616	8.470	8.534
140	2	4.713	4.544	4.619	4.401	4.175	4.449
220	3	.503	.469	.493	.471	.458	.495
320	4	1.094	1.000	1.050	1.022	1.033	1.087
430	5	.631	.572	.651	.644	.586	.630

* Model 5C

** From 20 x 20 Specimen

TABLE II. IDENTIFICATION OF GENERALIZED MASSES, 5 X 5 MODEL OF 20 X 20 SPECIMEN

Model	5A	5B	5C	5D	1**
Computer Experiment Number	296	297	292	295	-
Random Disp. Error	±5%	±5%	±5%	±5%	0
Bias Disp. Error	+5%	+5%	+5%	+5%	0
Random Error Seed	13	13	13	13	-
Mode	Generalized Masses ($Lb\text{-}Sec^2/In.$)				
1	8.544	8.538	8.543	8.568	8.534
2	4.506	4.506	4.619	4.610	4.449
3	.494	.494	.494	.493	.495
4	1.048	1.047	1.050	.994	1.087
5	.653	.653	.651	.629	.630

** From 20 x 20 Specimen

TABLE III. IDENTIFICATION OF GENERALIZED MASSES, 9 X 9 MODEL* OF 20 X 20 SPECIMEN

Computer Experiment Number		298	299	300	301	302	1**
Random Bias Error		0	±5%	±5%	±5%	±5%	0
Bias Disp. Error		0	+5%	+5%	+5%	+5%	0
Random Error Seed		-	5	13	421	1094	-
Station (In.)	Mode	Generalized Masses (Lb-Sec^2/In.)					
0	1	8.419	9.283	9.000	8.307	8.253	8.534
30	2	4.591	4.462	4.350	4.301	4.189	4.449
140	3	.504	.462	.472	.467	.483	.495
160	4	1.094	.975	1.042	1.053	1.095	1.087
220	5	.631	.659	.551	.577	.610	.630
280	6	.761	.717	.786	.674	.646	.743
340	7	1.213	1.152	1.154	1.208	1.052	1.177
400	8	1.439	1.371	1.401	1.322	1.370	1.412
460	9	.813	.713	.787	.860	.719	.786

* Model 9A

** From 20 x 20 Specimen

TABLE IV. IDENTIFICATION OF GENERALIZED MASSES, 9 X 9 MODEL OF 20 X 20 SPECIMEN

Model	9A	9B	9C	20 Pt
Computer Experiment Number	300	303	304	1*
Random Disp. Error	±5%	±5%	±5%	0
Bias Disp. Error	+5%	+5%	+5%	0
Random Error Seed	13	13	13	-
Mode	Generalized Masses ($Lb\text{-}Sec^2/In.$)			
1	9.000	9.015	9.043	8.534
2	4.350	4.335	4.513	4.449
3	.472	.472	.472	.495
4	1.042	1.042	1.138	1.087
5	.551	.549	.584	.630
6	.786	.783	.723	.743
7	1.154	1.243	1.120	1.177
8	1.401	1.411	1.396	1.412
9	.787	.708	.791	.786

* From 20 x 20 Specimen

TABLE V. IDENTIFICATION OF GENERALIZED MASSES, 12 X 12 MODEL* OF 20 X 20 SPECIMEN

Computer Experiment Number		305	306	312	307	308	1**
Random Disp. Error		0	±5%	±5%	±5%	±5%	0
Bias Disp. Error		0	+5%	+5%	+5%	+5%	0
Random Error Seed		-	5	13	421	1094	-
Station (In.)	Mode	Generalized Masses ($Lb\text{-}Sec^2/In.$)					
0	1	8.435	9.234	8.474	8.886	7.846	8.534
30	2	4.600	4.217	4.556	4.455	4.183	4.449
60	3	.504	.481	.488	.476	.432	.495
120	4	1.094	1.030	1.150	1.004	1.059	1.087
140	5	.631	.596	.596	.595	.616	.630
180	6	.761	.686	.722	.757	.741	.744
220	7	1.212	1.142	1.182	1.067	1.218	1.177
260	8	1.429	1.299	1.232	1.331	1.290	1.412
300	9	.813	.830	.797	.805	.790	.786
340	10	.169	.053	1.203	.265	.565	.043
400	11	.112	.091	.093	.102	.120	.172
460	12	1.135	1.070	1.177	.940	1.085	1.050

* Model 12B

** From 20 x 20 Specimen

TABLE VI. IDENTIFICATION OF GENERALIZED MASSES, 12 X 12 MODEL OF 20 X 20 SPECIMEN

Model	12B	12F	12A	20 Pt
Computer Experiment Number	312	311	309	1*
Random Disp. Error	±5%	±5%	±5%	0
Bias Disp. Error	+5%	+5%	+5%	0
Random Error Seed	13	13	13	-
Mode	Generalized Masses ($Lb/Sec^2/In.$)			
1	8.474	8.464	8.518	8.534
2	4.556	4.510	4.492	4.449
3	.488	.487	.487	.495
4	1.150	1.151	1.103	1.087
5	.596	.597	.595	.630
6	.722	.724	.777	.744
7	1.182	1.113	1.159	1.177
8	1.232	1.242	1.215	1.412
9	.797	.743	.789	.786
10	1.203	1.043	-.564	.043
11	.093	.104	.0103	.172
12	1.177	1.119	1.147	1.050

* From 20 x 20 Specimen

TABLE VII. MODEL DESCRIPTION

	Stations Used																			
Model	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5A	x					x				x					x					x
5B	x					x					x					x				x
5C	x					x				x					x				x	
5D		x				x						x			x				x	
9A	x	x				x	x			x			x			x		x		x
9B	x		x			x		x			x			x			x		x	x
9C		x	x			x	x			x			x			x		x		x
12A		x	x	x	x	x		x		x		x		x		x		x		x
12B	x	x	x		x	x		x		x		x		x		x		x		x
12F	x	x	x		x	x		x			x		x		x		x		x	

TABLE VIII. 20-POINT SPECIMEN DESCRIPTION

Sta No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Sta (In.)	0		60		120		160		200		240		280		320		370		430	
		30		100		140		180		220		260		300		340		400		460
Mass ($Lb\text{-}Sec^2/In.$)	.029		3.67		2.18		2.385		2.08		.910		.170			.070		.095	.210	
		1.05		3.71		2.18		2.59		1.56		.260		.085		.060		.120		.150
EI ($Lb\text{-}In.^2 \times 10^{10}$)	.35		.35		1.95		4.37		5.80		4.425		3.07		2.05		.975		.55	
		.35		1.20		3.00		5.70		5.60		3.6		2.60		1.60		.65		.50

Computer experiment 309 yielded a negative 10th generalized mass. All computer experiments that failed in this respect gave drastically unrealistic values of generalized mass. Ordinarily, using different stations or forcing frequencies produced proper identification of all modes.

RESPONSE FROM IDENTIFIED MODEL

Figures 1 through 12 portray typical real and imaginary acceleration mobility response obtained from the various models considered in the present study. In each instance, the exact curve represents the simulated experimental data for the twenty-point structure, obtained with zero error. Figures 1 and 2 provide the effect of random number seed for a typical five-point model. Figures 3 and 4 present the results obtained for one of the nine-point models considered in the investigation. Figures 5 and 6 show the effect of the random error seed on a twelve-point model. All computer experiments which incorporated error used a +5 percent random and a +5 percent bias on the real and imaginary displacement mobility data.

Figures 7, 8, 9, 10, 11 and 12 present the reidentified acceleration mobility, both real and imaginary, for typical five-, nine-, and twelve-point models respectively. The models varied in that different spanwise masses were considered. Some of the models employed in the study are given in Table VII showing the various points of interest for each model. For each model, the computer experiments were executed using the same random number seed and the aforementioned errors were incorporated. As evidenced by the figures, the various models provided acceptable reidentification of the twenty-point specimen simulated experimental displacement mobility data.

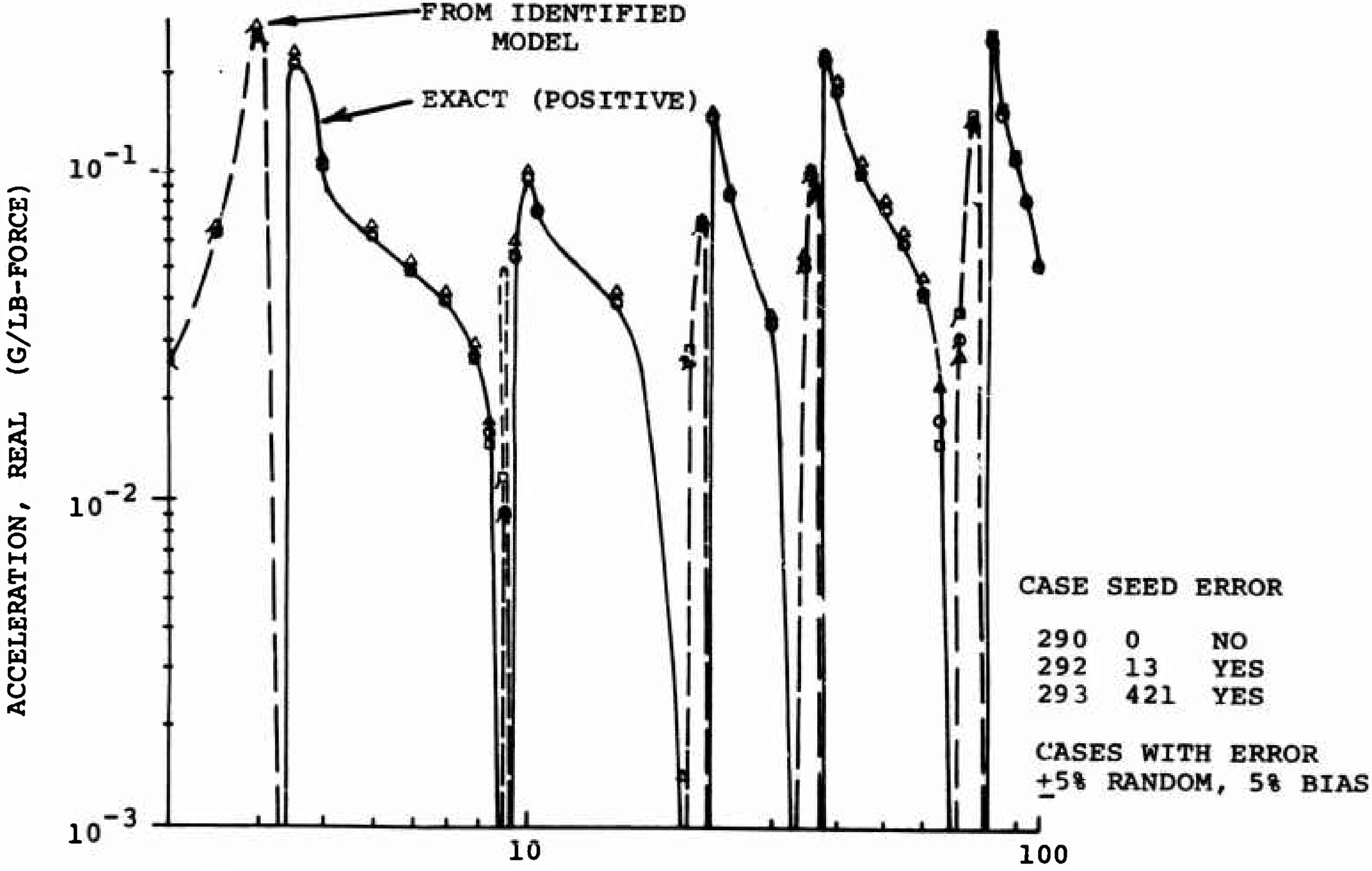

Figure 1. Effect of Error on Five-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

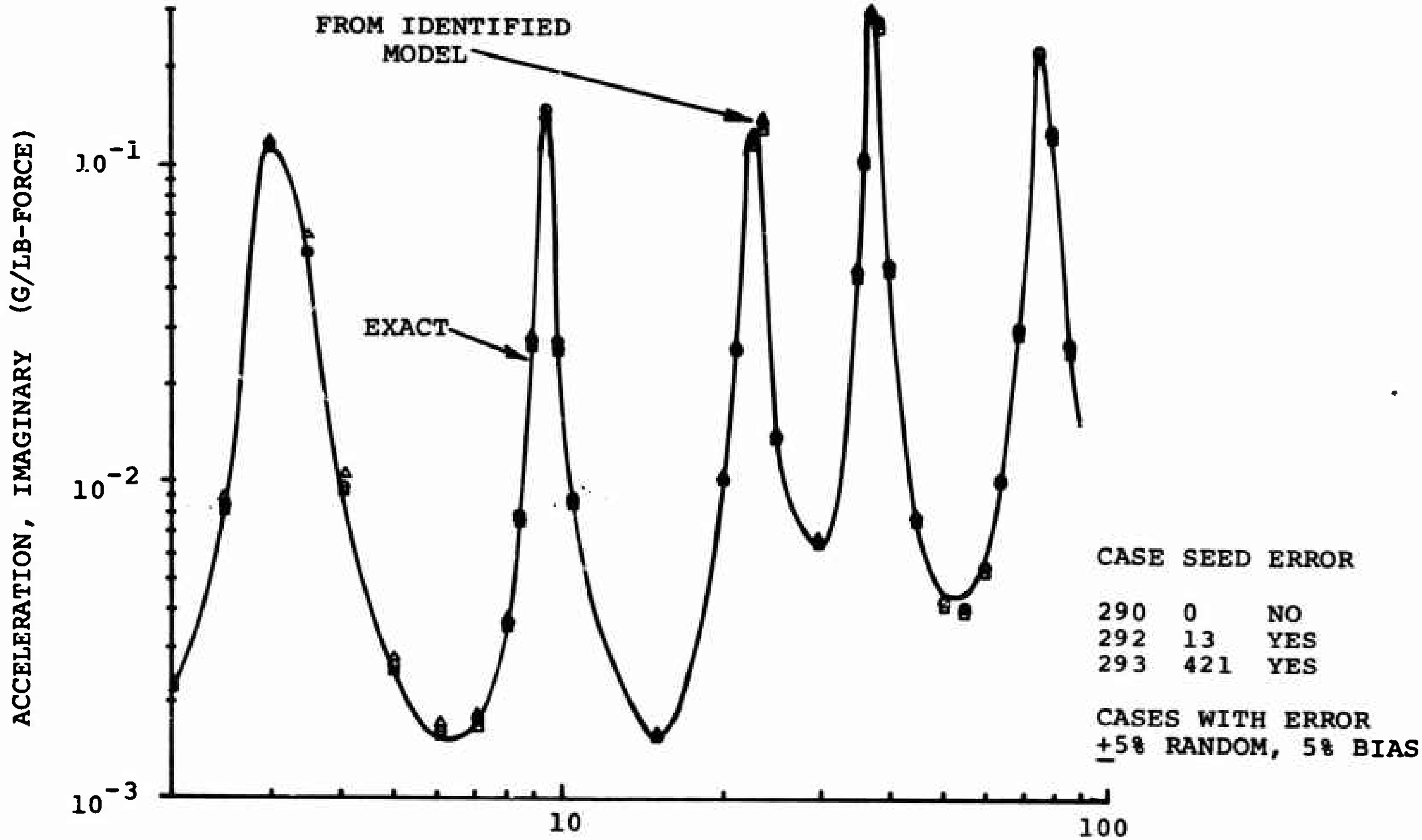

Figure 2. Effect of Error on Five-Point Model Identification of Imaginary Acceleration Response; Driving Point at Hub.

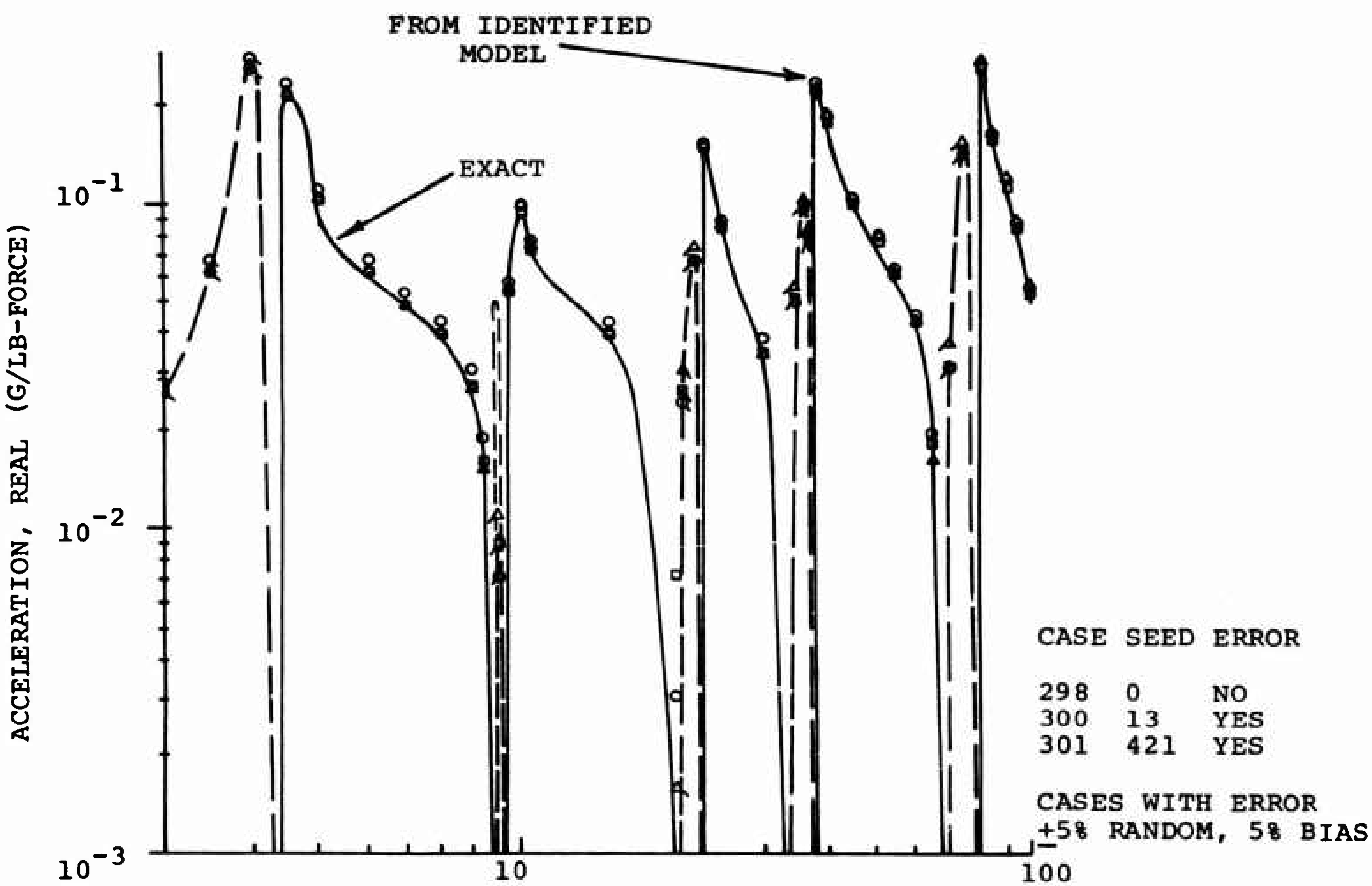

Figure 3. Effect of Error on Nine-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

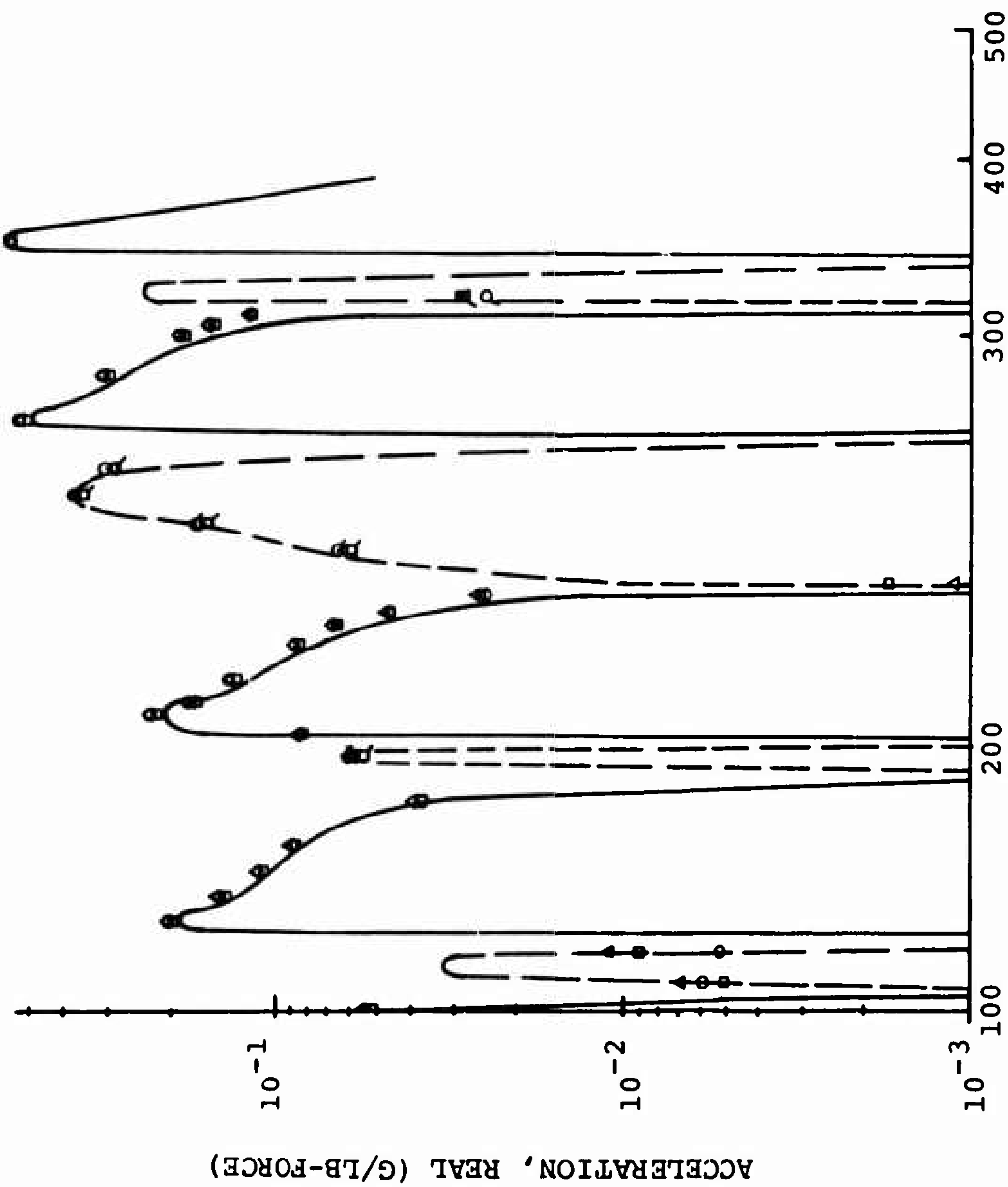

Figure 3 - Continued.

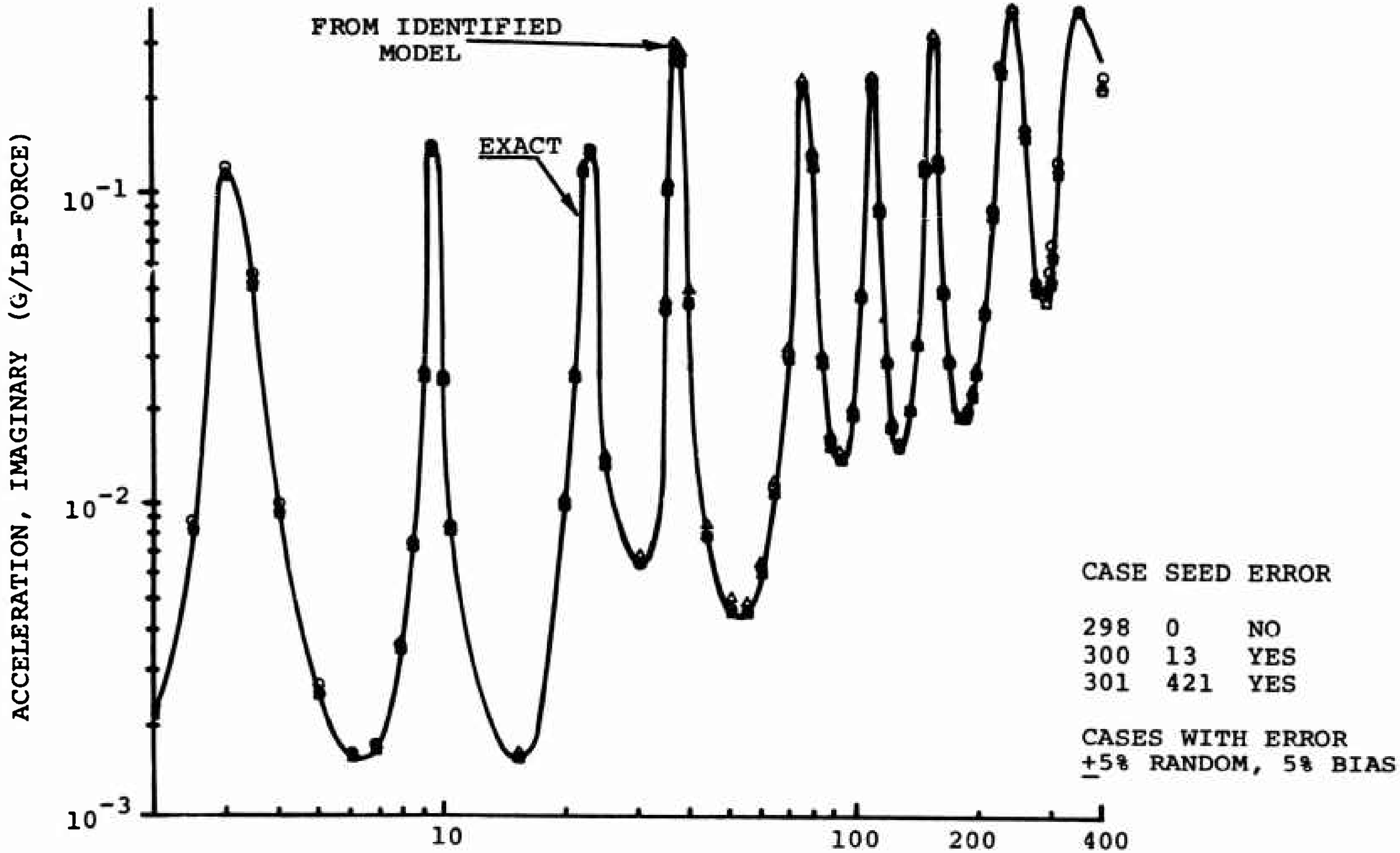

Figure 4. Effect of Error on Nine-Point Model Identification of Imaginary Acceleration Response; Driving Point at Hub.

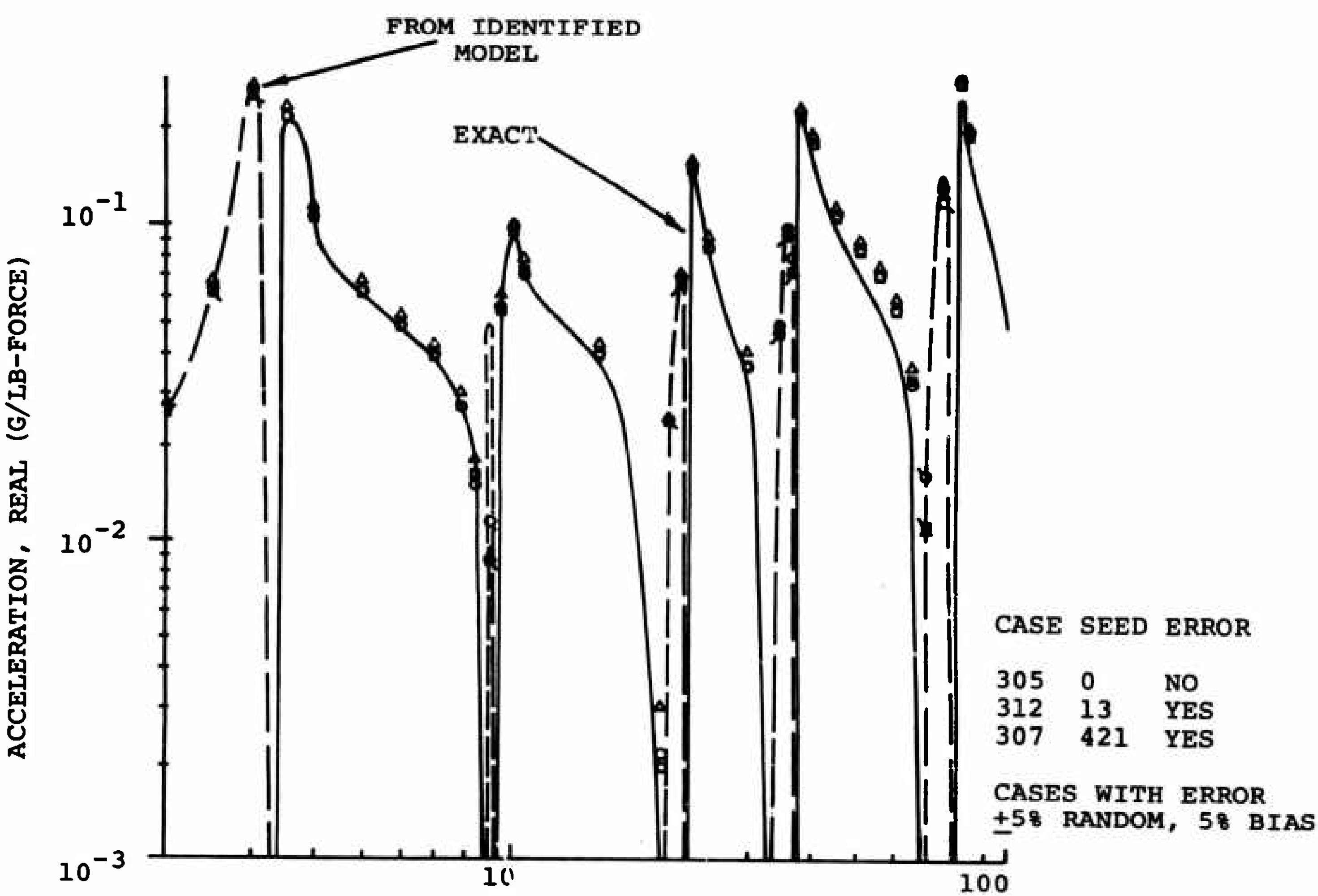

Figure 5. Effect of Error on Twelve-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

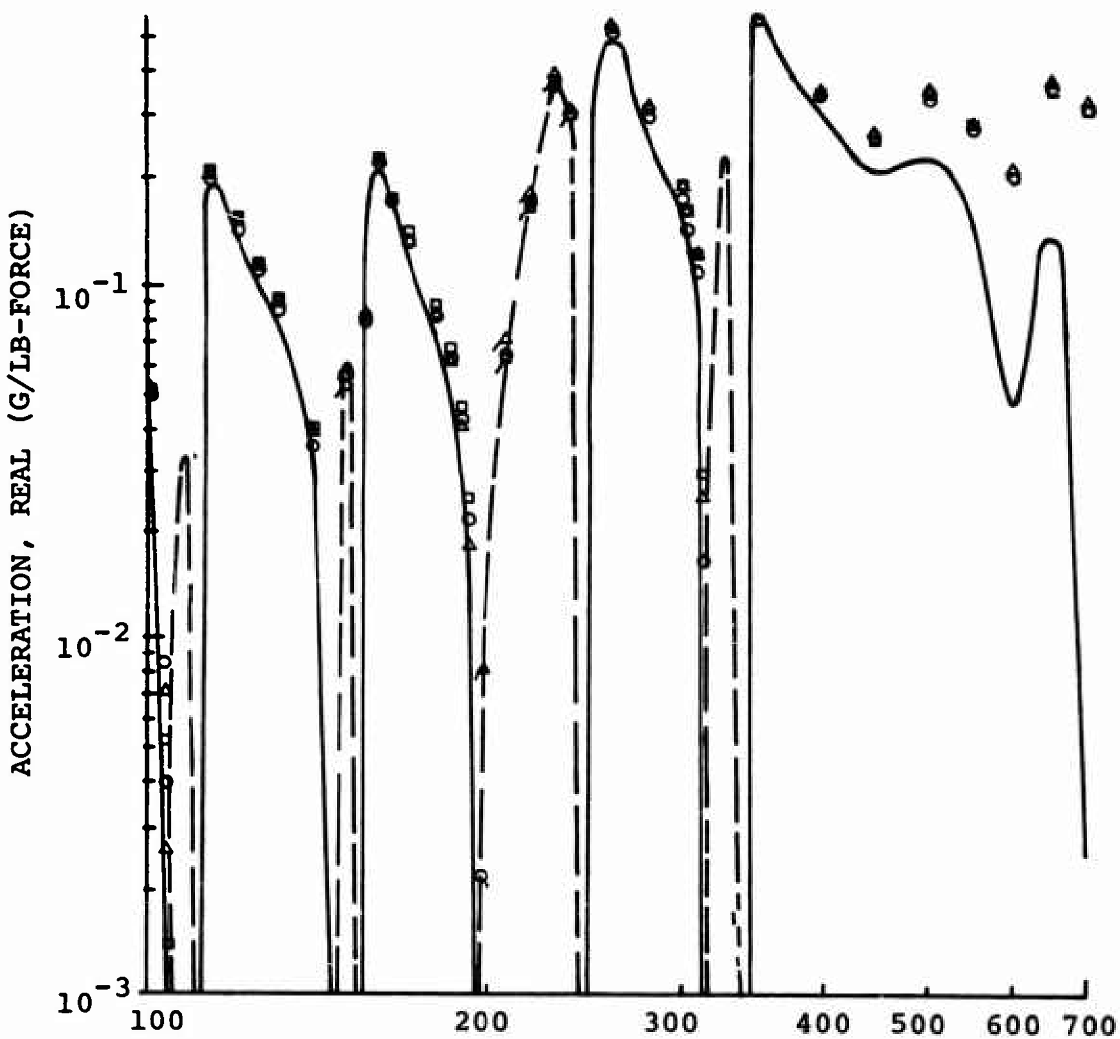

Figure 5 - Continued.

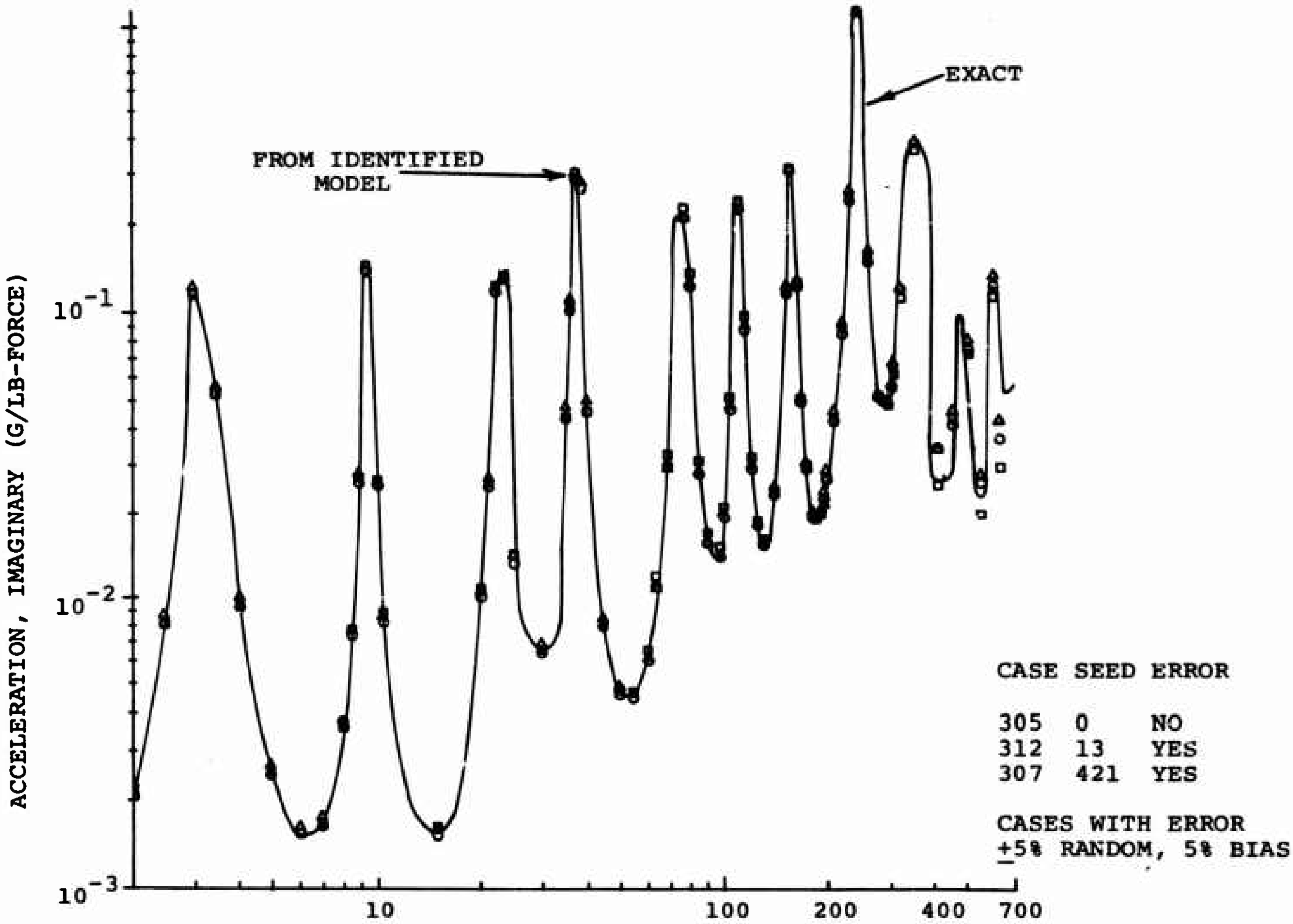

Figure 6. Effect of Error on Twelve-Point Model Identification of Imaginary Acceleration Response; Driving Point at Hub.

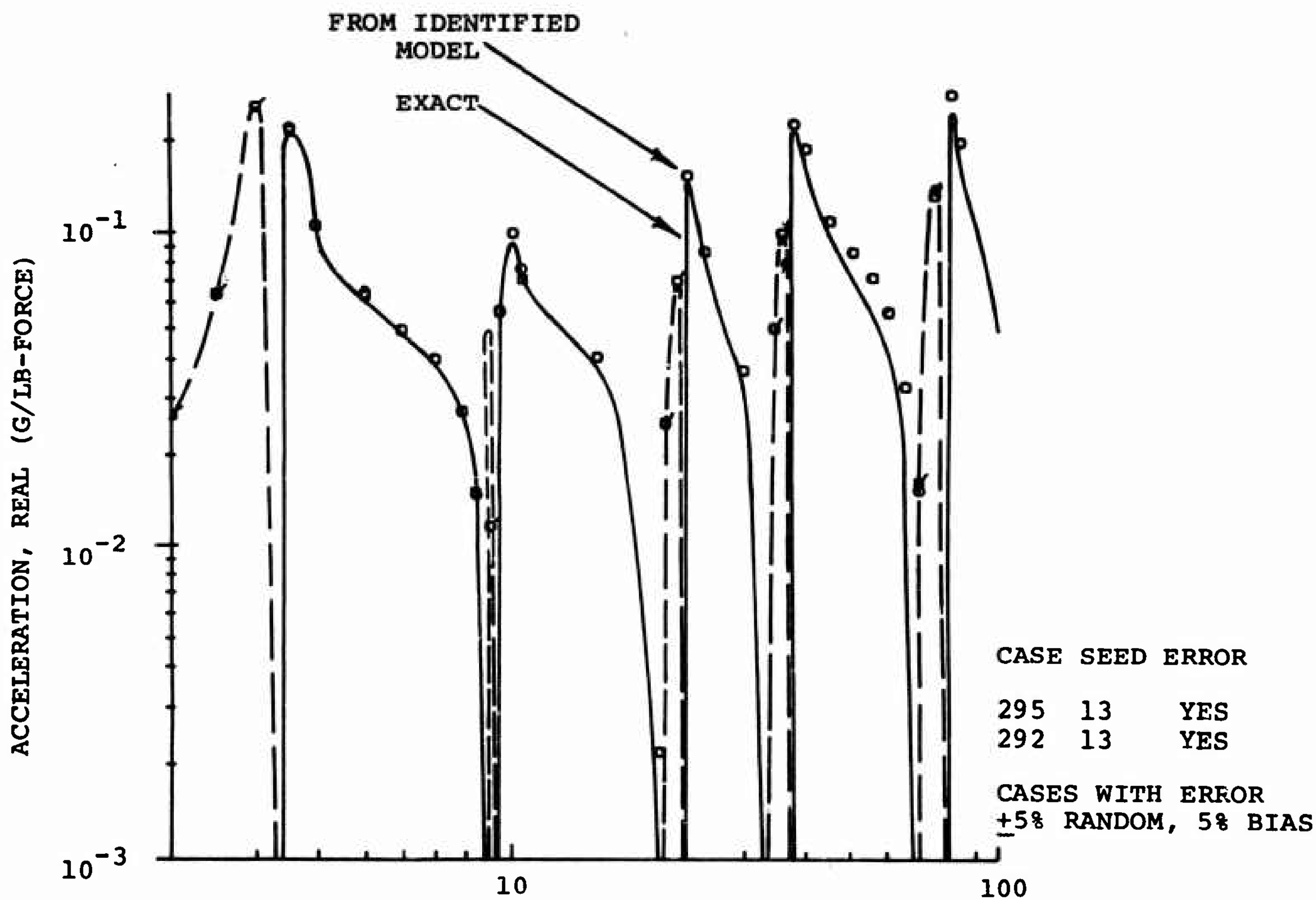

Figure 7. Effect of Model on Five-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

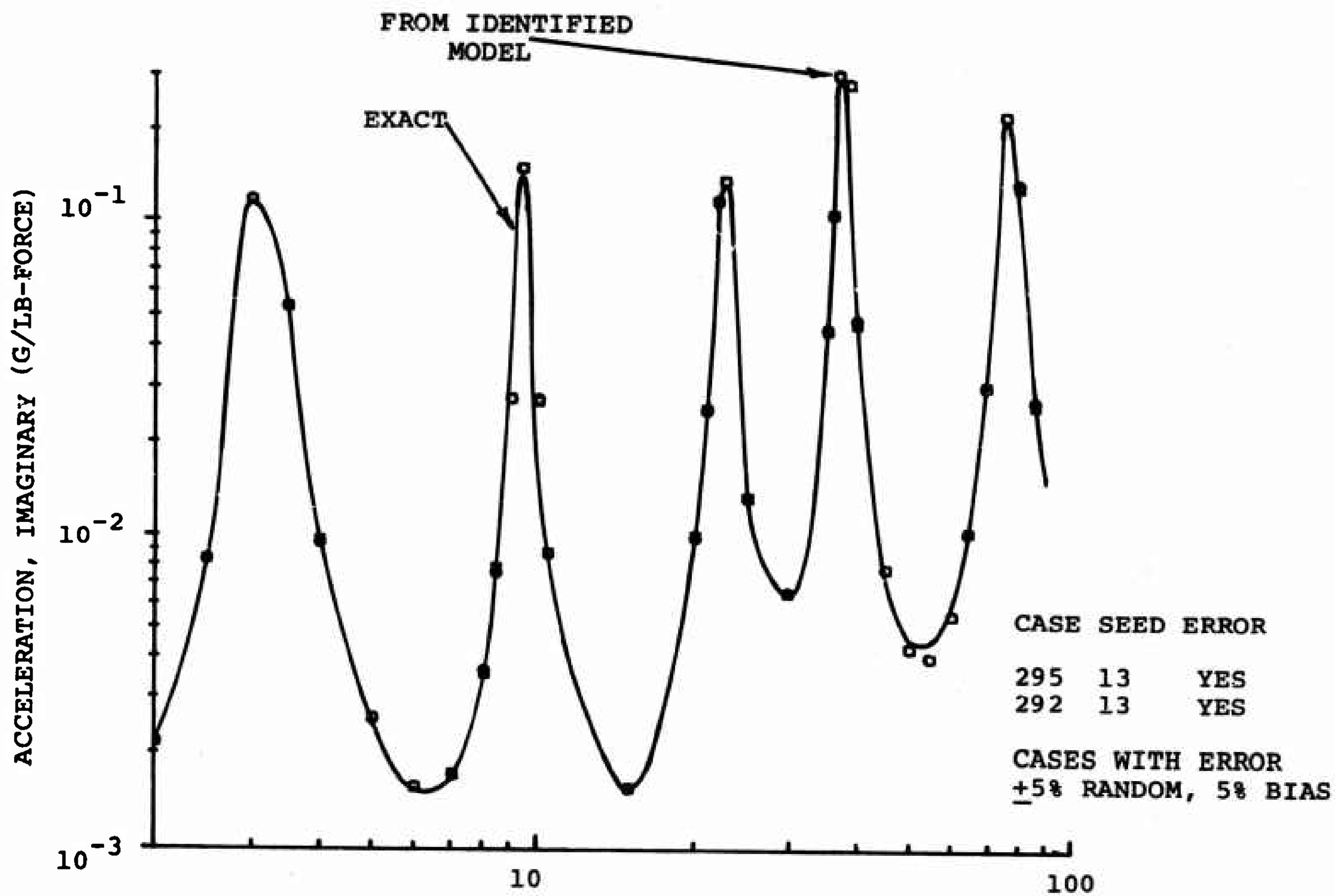

Figure 8. Effect of Model on Five-Point Model Identification of Imaginary Acceleration Response; Driving Point at Hub.

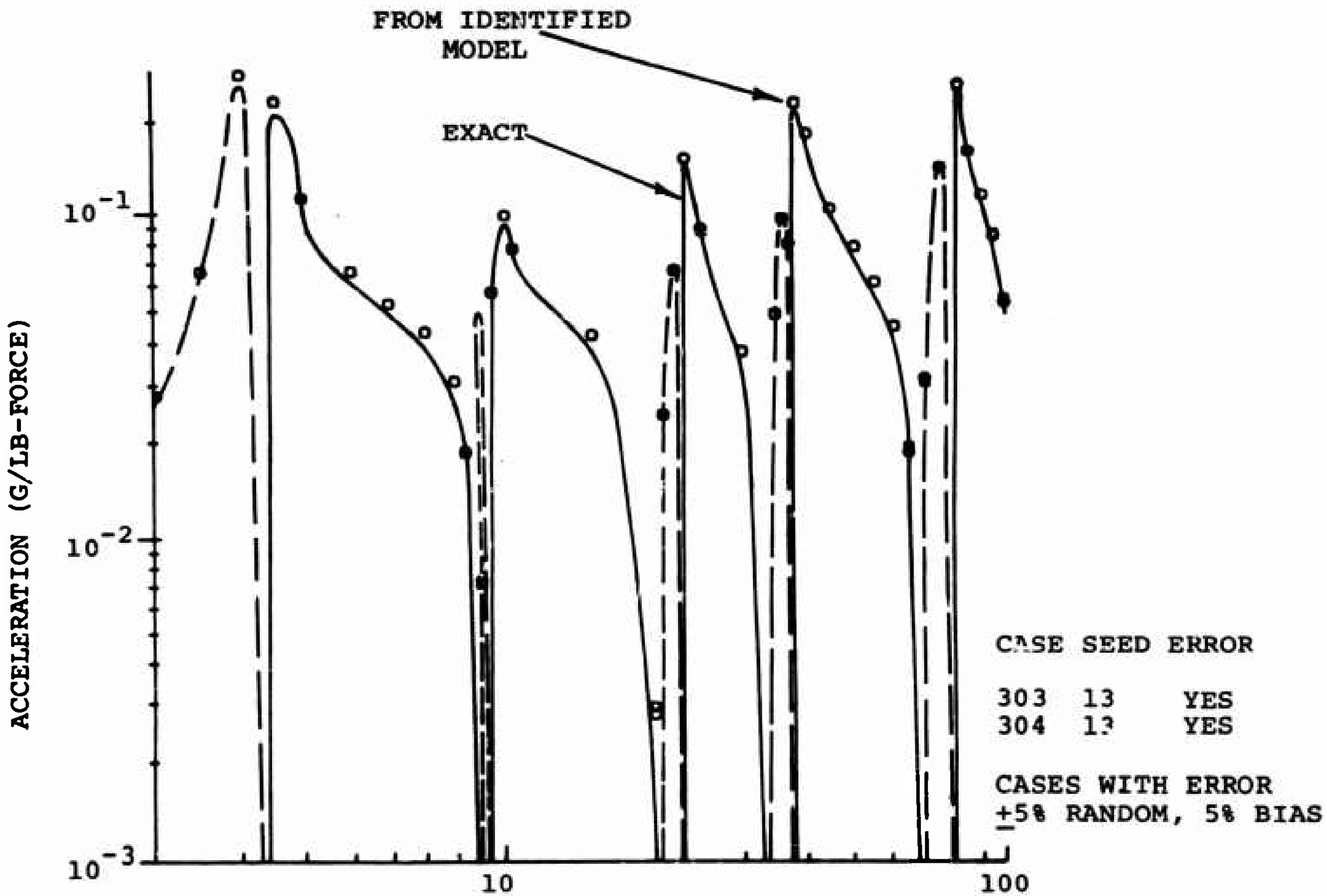

Figure 9. Effect of Model on Nine-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

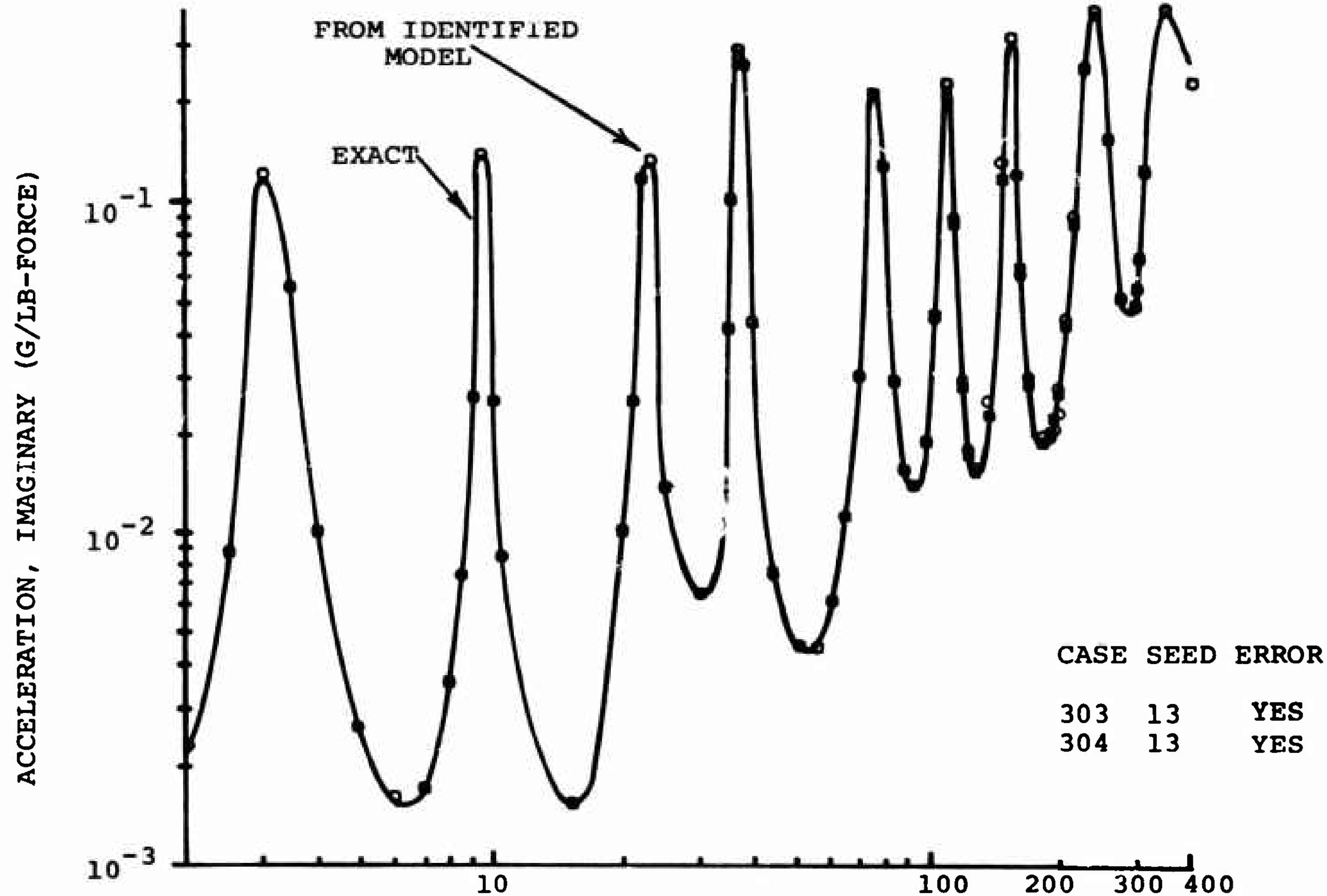

Figure 10. Effect of Model on Nine-Point Model Identification of Imaginary Acceleration Response; Driving Point at Hub.

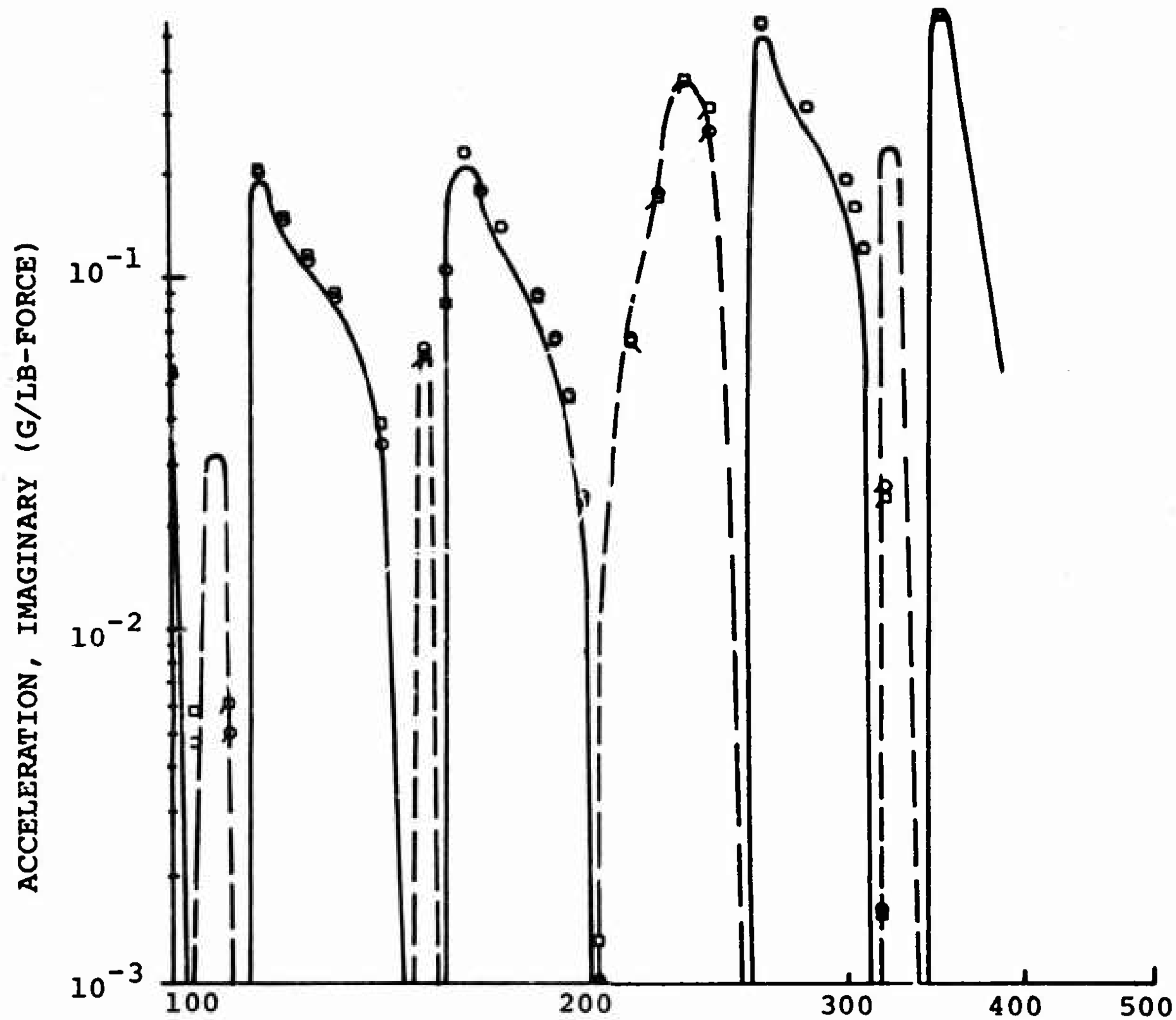

Figure 10 - Continued.

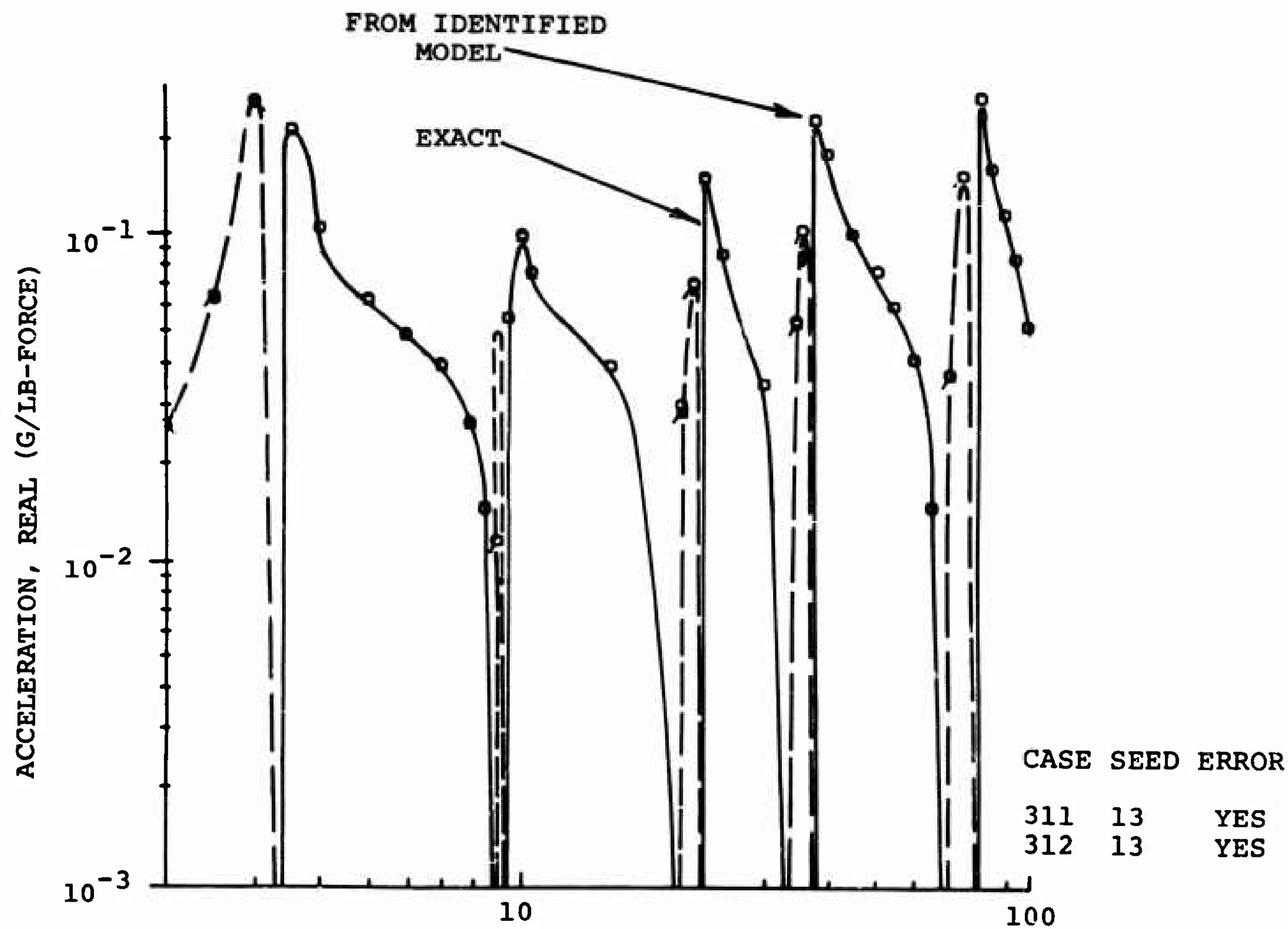

Figure 11. Effect of Model on Twelve-Point Model Identification of Real Acceleration Response; Driving Point at Hub.

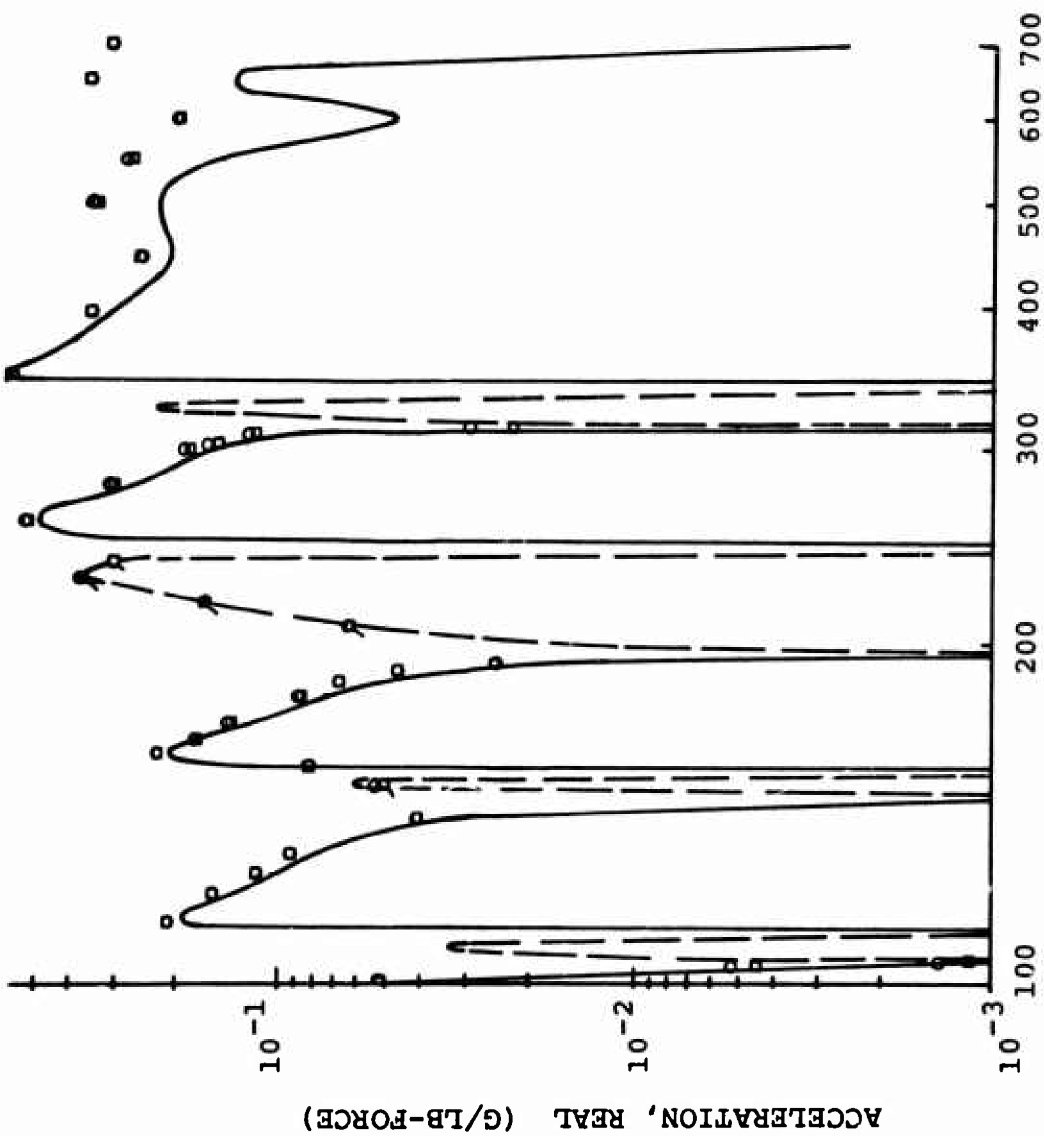

Figure 11 - Continued.

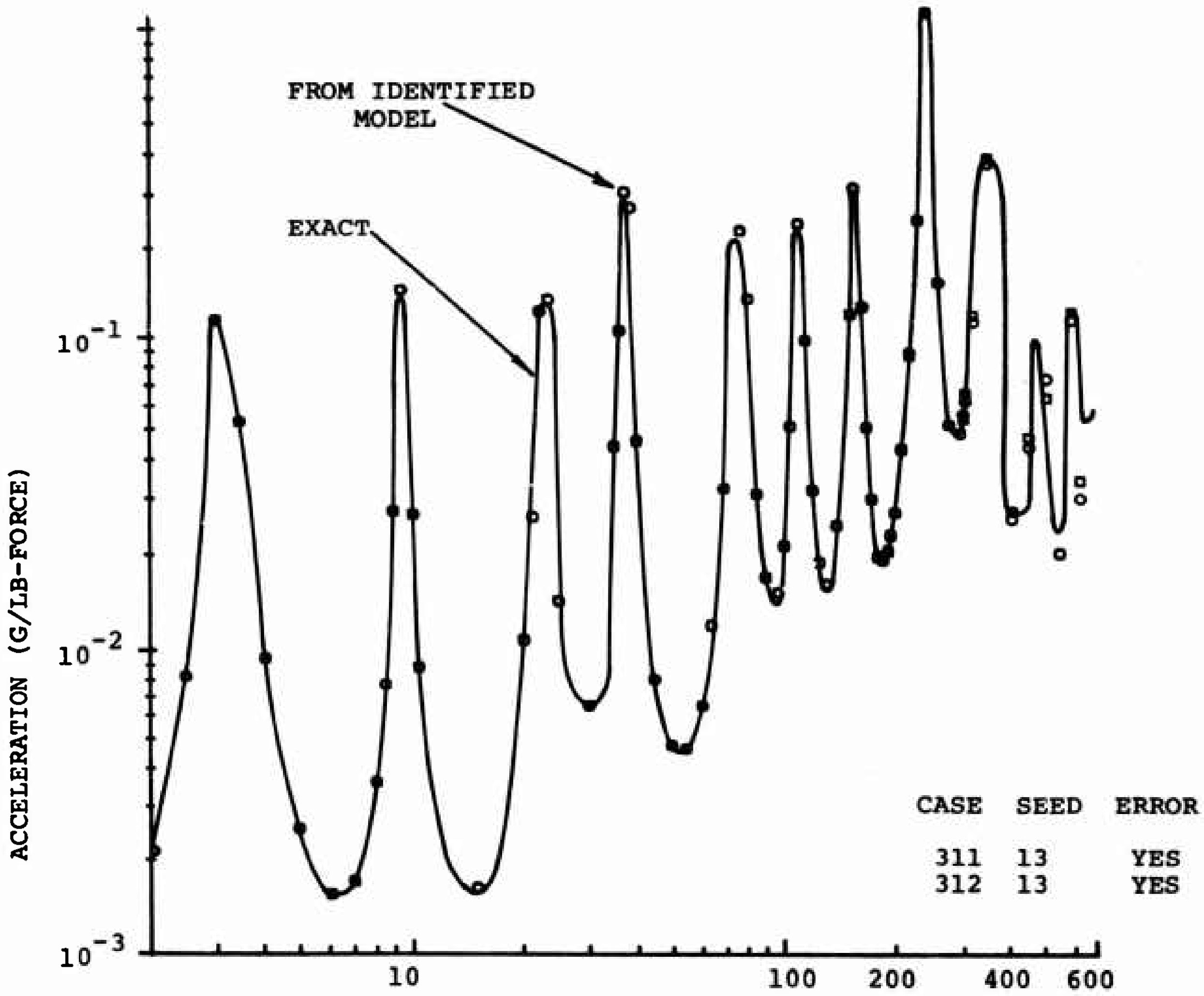

Figure 12. Effect of Model on Twelve-Point Model Identification of Imaginary Acceleration Response ; Driving Point at Hub.

CONCLUSIONS

1. Single-point excitation of a structure yields the necessary mobility data to satisfactorily determine the mass, stiffness and damping characteristics for a mathematical model having less degrees of freedom than the linear elastic structure it represents.

2. The method does not require an intuitive mathematical model and uses only a minimum amount of impedance-type test data.

3. The eigenvector or mode shape associated with each natural frequency is also determined in the analysis.

4. Computer experiments using simulated test data indicate the method is insensitive to the level of measurement error inherent in the state of the measurement art.

5. A fully populated mass matrix should be assumed for an accurate analytical model of a real structure.

LITERATURE CITED

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APPENDIX
COMPUTER PROGRAM DESCRIPTION

A digital computer program was designed for computer experiment to investigate the proper physical interpretation of identified parameters for use in helicopter engineering. The program was written for the IBM 360/40 operating system using FORTRAN IV language. A flow chart indicating the program logical procedure is shown in Figure 13. A description of the input cards and a program source listing are included in this appendix.

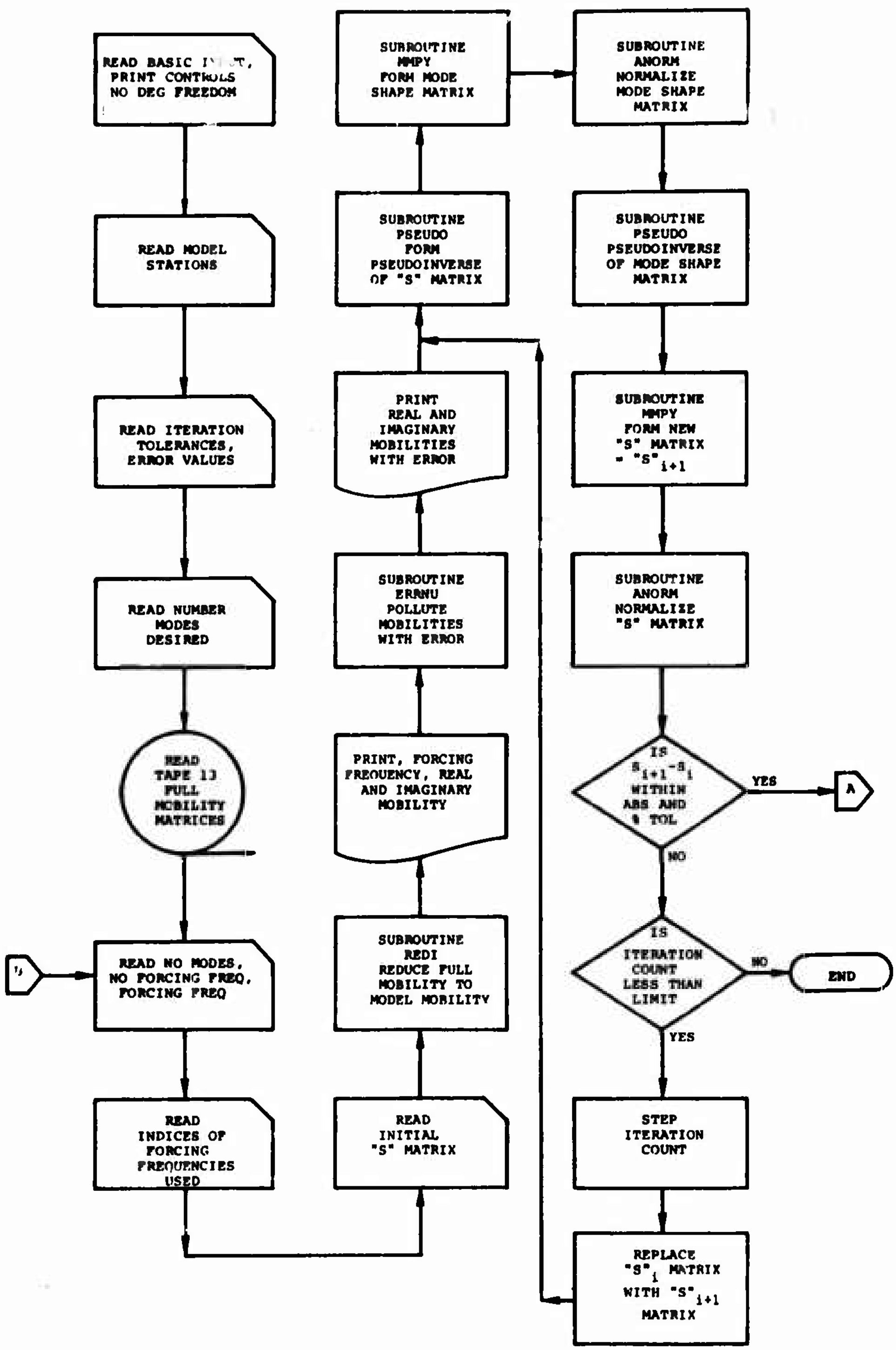

Figure 13. Flow Chart of Computer Program.

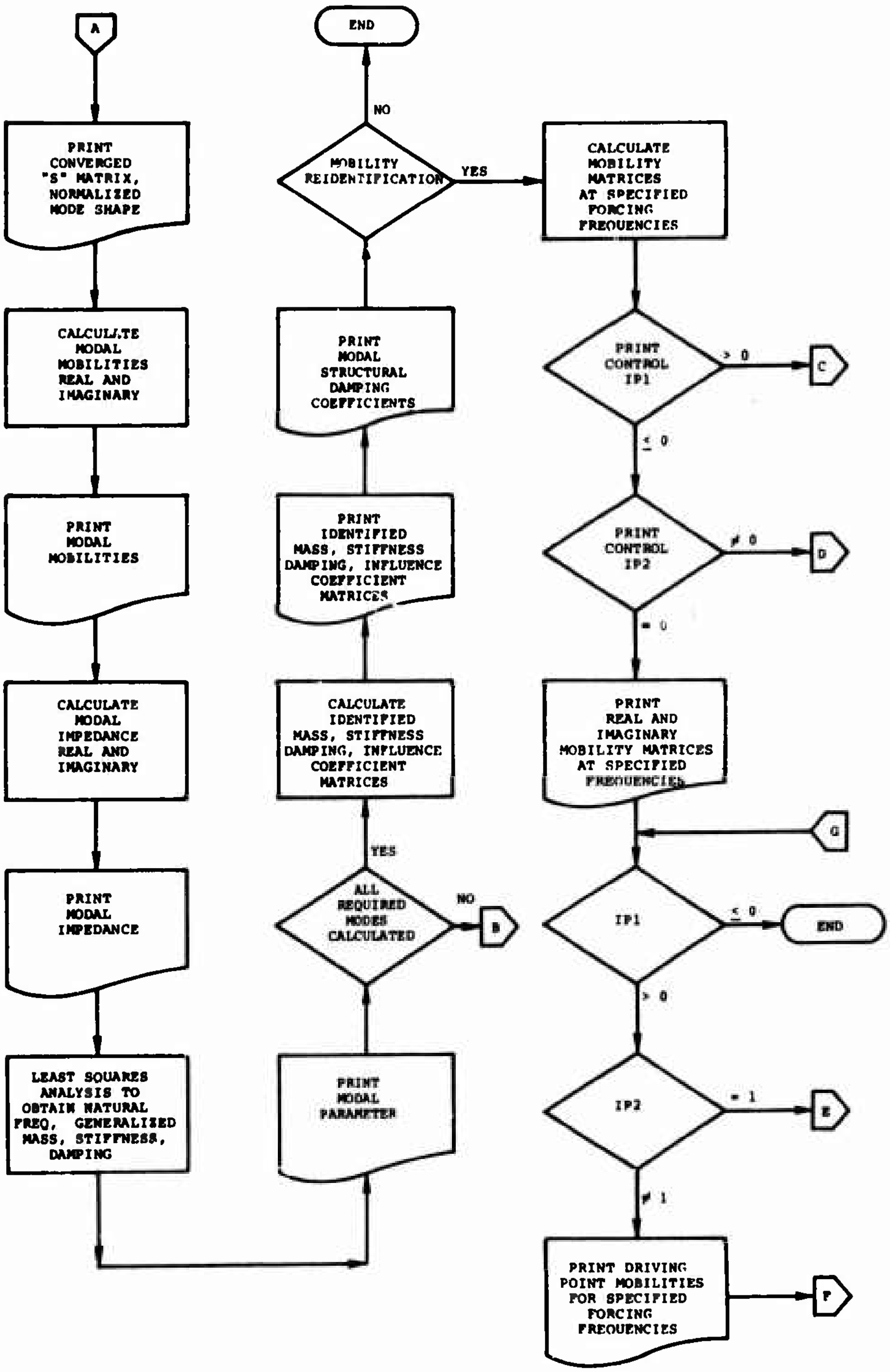

Figure 13 - Continued.

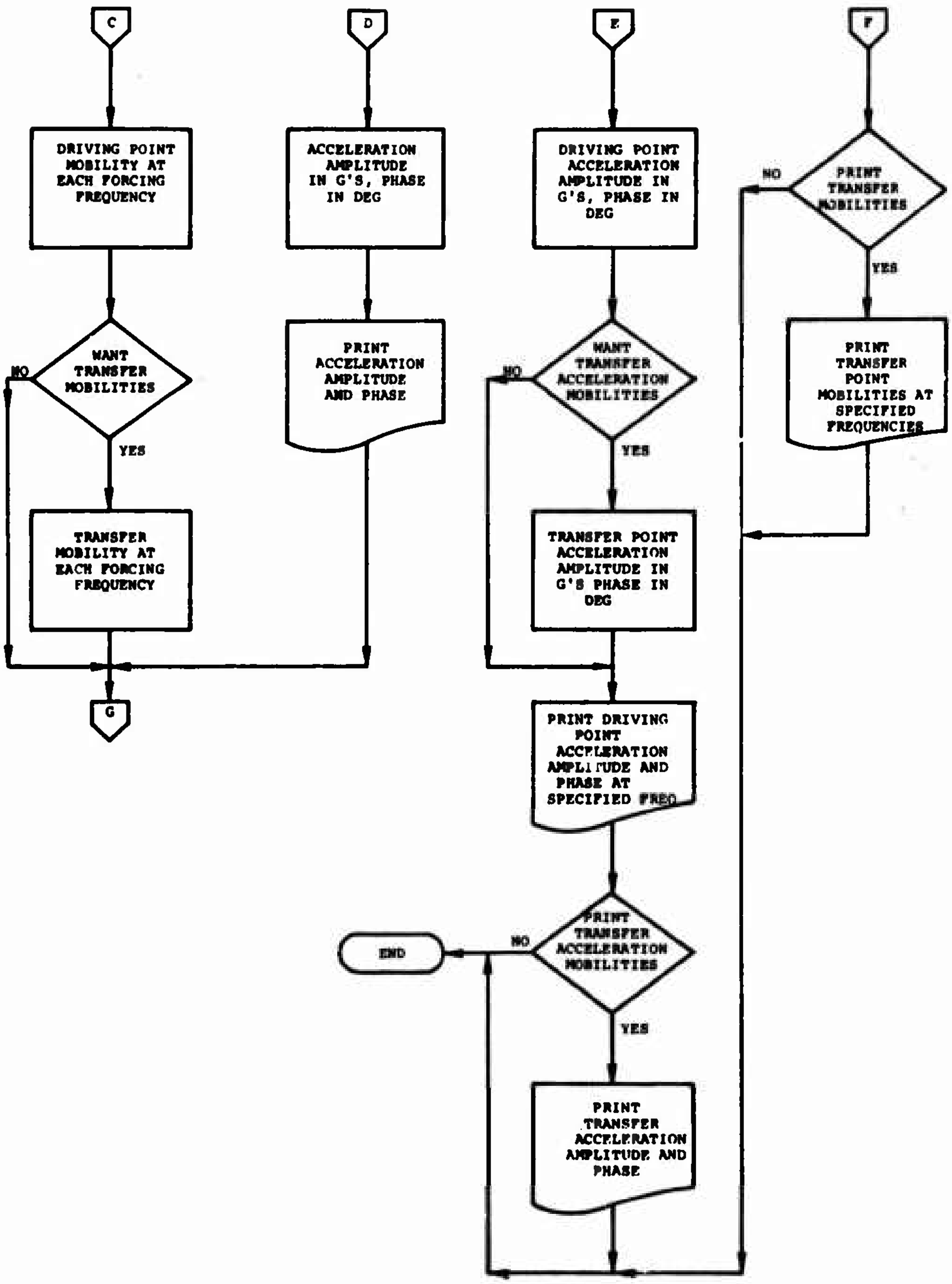

Figure 13 - Concluded.

DESCRIPTION OF INPUT CARDS

Note: All integer variables must be right justified with no decimal point.

Tape, Card Reader and Printer Assignments

1 Card Reader

3 Printer (On Line)

13 Tape Assignment. Contains displacement mobility data for all degrees of freedom, with no error for specified frequencies.

All input data must be in the following units:

Mass - Lb-Sec^2/In.

Stiffness - Lb/In.

Frequencies- Hz

INPUT STRUCTURAL DYNAMICS PROGRAM STIDN

Card No. 1	Columns 1-10	IP1	Control of Printed Output IP1=0 Print Full Mobility Matrix, Real and Imaginary at Each Specified Frequency IP1=1 Print Only Diagonal Elements and Row of Mobility Matrix, Real and Imaginary at Each Specified Frequency
	11-20	IP2	IP2=1 Print Full Acceleration Amplitude in G's and Phase Angle in Degrees at Each Specified Frequency IP2=2 Print Only Diagonal Elements and Row of Acceleration Amplitude in G's and Phase Angle in Degrees at Each Specified Frequency
	21-30		NROW Row of Displacement Mobilities or Acceleration Amplitudes to be Printed When IP2=2
	31-40		NN Control on Type of Damping Used in Re-identification of Mobilities NN = 0 Use Scalar Structural Damping Coefficient x K Matrix NN = 1 Use Damping Matrix
	41-50	NJ	Number of Points Tested (Number of Degrees of Freedom)
	51-60	NK	Number of Force Input Station

Card No. 1 (Contd)	Columns 61-70	ITMS	Limit on Number of Mode Shape Iterations
	71-80	NFF	Number of Frequencies at Which Reidentification of Mobilities is Calculated
Card No. 2		KEEP	Stations to be Used in Model. Ten Columns Per Value Maximum of 8 Values Per Card (Format 8I10)
Card No. 3	1-10	ATOL	Absolute Tolerance Used in Mode Shape Iteration
	11-20	PTOL	Percentage Tolerance Used in Mode Shape Iteration
	21-30	PCTR	Random Error Applied to Real Mobilities, Uniform Between - And + PCTR
	31-40	PCTBR	Bias Error Applied to Real Mobilities
	41-50	PCTI	Random Error Applied to Imaginary Mobilities Uniform Between - And + PCTI
	51-60	PCTBI	Bias Error Applied to Imaginary Mobilities
	61-70	IZ	Random Number Seed
	71-80	IA	Print Control IA = 0 Displacement Mobilities Printed IA ≠ 0 Acceleration Mobilities Printed
Card No. 4	1-10	NPHI	Number of Modes Desired

The following cards (5-8 inclusive) are repeated NPHI Times

Card No. 5	Columns 1-10	NQ	Number of Modes to be Calculated at Each Natural Frequency (Usually 2 or 3)
	11-20	NP	Number of Forcing Frequencies Used in Calculating the Number of Modes
Card No. 6		OMF	Forcing Frequencies Used in Calculating the NQ Modes (NP Forcing Frequencies). Ten Columns Per Value, 8 Values Per Card. Format (8F10.4). Hertz
Card No. 7		INDX	The Number of Each Forcing Frequency Used. (Frequencies are Stored on Tape 13)
Card(s) No. 8		S	Matrix Used in Iteration for Mode Shape (Format 8F10.4)
Card(s) No. 9		HZ	Frequencies at Which Reidentification of Mobilities is to be Calculated. Ten Columns Per Value, 8 Values Per Card (Format 8F10.4) Hertz
Card No. 10	1-10	IC	Control on Subsequent Cases

```
C     STRUCTURAL DYNAMICS  ---  SINGLE POINT FORCING
C
C
      INTEGER HEAD(20),HT(7)
      DIMENSION  S(20,21),YI(20,21),GSQ(20),
     AYR(20,21 ),PHI(20,21),PHIA(20,21),OMF(20),G(20),SF(20,21),
     BSM(20,21),SI(20,21),ZSI(20,21),
     CAKS(20),AMS(20),ADS(20),OMN(20),YRT(20,100),YIT(20,100)
      DIMENSION  PHIN(20,21),YIN(20,21),PHIM(20,21)
      DIMENSION  ZSR(20,21),OMFS(20),OMFR(20),TR(100,20),TI(100,20)
      DIMENSION OMNC(20),AKSR(20),AMSR(20),OMNS(20),ADSR(20)
      DIMENSION HZ(100),INDX(20 ),KEEP(20),PHIT(20,21)
      DIMENSION DPR(100,20),DPI(100,20),ZR(20,21),ZI(20,21)
      LOGICAL TORF
      DATA HT/'EXEC','T DA','TA S','IMUL','ATED',' TES','T  '/
C     DISPLACEMENT MOBILITY DATA ON TAPE 13 (184)
C     NJ=NUMBER OF GENERALIZED COORDINATES
C     NUMBER OF DEGREES OF FREEDOM
C     NQ=NUMBER OF MODES
C     NP=NUMBER OF FORCING FREQUENCIES
C     NK=FORCE INPUT STA
      REWIND 14
      READ (1,140) IP1,IP2,NROW,NN,NJ,NK,ITMS,NFF
      READ (1,140) (KEEP(I),I=1,NJ )
  100 READ (1,120) ATOL,PTOL,PCTR,PCTBR,PCTI,PCTBI,IZ,IA
      IX=IZ*2+1
      READ (1,140 ) NPHI
      REWIND 13
      READ (13) NCOL,HT,HEAD,NF,ND,( HZ(I),I=1,NF )
      DO 110 L=1,NF
  110 READ (13) HZ(L),((YRT(I,L),YIT(I,L)),I=1,ND )
  120 FORMAT (6F10.4,2I10 )
      WRITE (3,130) PCTR,PCTBR,PCTI,PCTBI,IZ
  130 FORMAT ('1',T10,'MAX RAND ERROR ON REAL      ='F6.3,10X,'BIAS ERROR
     A ON REAL      ='F6.3,' OF ELEMENTS'/T10,'               ON IMAGINAR
     BY='F6.3,21X,'ON IMAGINARY='F6.3,15X,'SEED='I5///)
  140 FORMAT (8I10)
  150 FORMAT (8F10.4)
      WRITE (3,170)  (KEEP(I),I=1,NJ )
      WRITE (3,160) KEEP(NK)
  160 FORMAT (/////'             FORCE INPUT IS AT STA 'I3///)
  170 FORMAT (/////T10,'STATIONS USED '//(T20,10I5))
      PTOL=PTOL/100.
      DO 480 MM=1,NPHI
      READ (1,140) NQ,NP
      READ (1,150) (OMF(I),I=1,NP)
      DO 180 I=1,NP
      OMFR(I)=OMF(I)*6.283185
  180 OMFS(I)=OMFR(I)*OMFR(I)
      READ (1,140) (INDX(I),I=1,NP )
C     READ INITIAL S MATRIX (ROWWISE)
      DO 190 I=1,NQ
  190 READ (1,150) (  S(I,J),J=1,NP)
      CALL REDI (YR,YI,NP,NJ,KEEP,INDX ,YRT,YIT )
      ITC=1
```

```
      WRITE (3,200) (CMF(I),I=1,NP)
  200 FORMAT (//////////T50, 'FORCING FREQUENCIES '//(10F12.4)/////)
      WRITE (3,210)
  210 FORMAT ('1',T50,'REAL MOBILITY MATRIX'//)
      CALL MOUT2 (YR,NJ,NP )
      WRITE (3,220)
  220 FORMAT ('1',T50, 'IMAGINARY MOBILITY MATRIX'//)
      CALL MOUT2 (YI,NJ,NP )
      IF(PCTR.NE.0.OR.PCTBR.NE.0.OR.PCTI.NE.0.OR.PCTBI.NE.0 ) CALL ERRNU
     A (YR,YI,PCTR,PCTBR,PCTI,PCTBI,NJ,NP,IX )
      WRITE (3,230)
  230 FORMAT ('1',T50,'MOBILITY MATRICES WITH ERROR   REAL,IMAGINARY')
      CALL MOUT2 (YR,NJ,NP )
      CALL MOUT2 (YI,NJ,NP )
C
C          NORMALIZE IMAGINARY MOBILITY
C
C
C
C     ITERATE FOR MODE SHAPE AND S MATRIX
  240 CALL PSEUDO (S,NQ,NP,SM )
      WRITE (3,250) ITC
  250 FORMAT ('    S ITERATION='I4)
      CALL MOUT2 (S,NQ,NP )
C
C
      CALL MMPY ( YI ,SM,NJ,NP,NQ,PHI )
C
C
  260 FORMAT (///'  PHI MATRIX'/)
C
C
C
C          NORMALIZE PHI MATRIX
      CALL ANORM (PHI,PHIM,NJ,NQ )
      CALL PSEUDO (PHI,NJ,NQ,PHIA )
C
C
C
C
C
  270 CALL MMPY ( PHIA,YI ,NQ,NJ,NP,SI )
C
C
      CALL TRAN (  SI,SM,NQ,NP )
      CALL ANORM (SM,ST,NP,NQ )
      CALL TRAN (ST,SI,NP,NQ )
C     CHECK CONVERGENCE OF S MATRIX
      DO 300 I=1,NQ
      DO 300 J=1,NP
      DEL= SI(I,J)-S(I,J)
      IF (ABS(DEL)-ATOL )  300,300,280
  280 IF (S(I,J)) 290,310,290
  290 IF (ABS(DEL/S(I,J))-PTOL)    300,300,310
  300 CONTINUE
```

```
      GO TO 360
  310 IF (ITC-ITMS ) 320,320,340
  320 ITC=ITC+1
      DO 330 J=1,NP
      DO 330 I=1,NQ
  330 S(I,J)= SI(I,J)
      GO TO 240
  340 WRITE  (3,350)
  350 FORMAT (T10,'MAXIMUM NUMBER OF S MATRIX ITERATIONS EXCEEDED,
     *JOB TERMINATED')
      GO TO 870
  360 WRITE (3,260)
      CALL MOUT2 ( PHIM,NJ,NQ )
      WRITE (3,370 )
  370 FORMAT ('   CONVERGED S MATRIX'/)
      CALL MOUT2 (SI,NQ,NP )
C     CALCULATE MODAL MOBILITY
C      SM=Y*  REAL       SI=Y*  IMAG
      CALL PSEUDO (PHIM,NJ,NQ,PHIN )
      CALL MMPY (PHIN,YR,NQ,NJ,NP, SM )
      CALL MMPY (PHIN,YI,NQ,NJ,NP, SI )
      WRITE (3,380)
  380 FORMAT ('1',T10,'MODAL MOBILITIES,  REAL,  IMAGINARY'//)
      CALL MOUT2 ( SM,NQ,NP )
      CALL MOUT2 ( SI,NQ,NP )
C
C
C     CALCULATE MODAL IMPEDANCE
C
      DO 390 I=1,NQ
      WRITE (3,150 ) PHIM(NK,I )
      DO 390 J=1,NP
      CON=PHIM(NK,I)/(SI(I,J)* SI(I,J)+ SM(I,J)* SM(I,J))
      ZSR(I,J)= SM(I,J)*CON
  390 ZSI(I,J)=- SI(I,J)*CON
      WRITE (3,400)
  400 FORMAT  ('1',T10,'MODAL IMPEDANCE    REAL,IMAGINARY'//)
      CALL MOUT2 (ZSR ,NQ,NP )
      CALL MOUT2 (ZSI,NQ,NP )
C
C
C     LEAST SQUARES ANALYSIS ON MODAL IMPEDANCE AS FUNCTION
C     OF FORCING FREQUENCY SQUARED
C
C
      NL=NP/NQ
      ANL=NL
      NLC=NL
      KJ=1
      DO 420 K=1,NQ
      SUM =0.
      SUMA=0.
      SUMB=0.
      SUMC=0.
      DO 410 I=KJ,NLC
```

```
      SUM  =OMFS(I)+SUM
      SUMA=ZSR(K,I)+SUMA
      SUMB=OMFS(I)*OMFS(I)+SUMB
  410 SUMC =OMFS(I)*ZSR(K,I)+SUMC
      DET=ANL*SUMB-SUM*SUM
      XA=(SUMA*SUMB-SUMC*SUM)/DET
      XB=(ANL*SUMC-SUMA*SUM)/DET
      KJ=NLC+1
      NLC=NL*(K+1)
      OMNC(K)=SQRT(ABS (XA/XB ))
      AKSR(K)=-XB*OMNC(K)*OMNC(K)
      AMSR(K)=-XB
      OMNS(K)=OMNC(K)*OMNC(K)
  420 CONTINUE
      L=1
      DO 430 I=1,NQ
      ADSR(I)=(OMFS(L)/OMNS(I)-1.)* SI(I,L)*AKSR(I)/ SM(I,L)
      OMNC(I)=OMNC(I)/6.28318
  430 L=2*I+1
C
C
      IF (  MM.NE.1 ) GO TO 450
      SUM=0.
      DO 440 I=1,NL
  440 SUM=ZSI(   1,I)+SUM
      G(   1 )=SUM/(AKSR(   1 )*ANL)
      OMN(1)=OMNC(1)
      ADS(1)=ADSR(1)
      AMS(1)=AMSR(1)
      AKS(1)=AKSR(1)
      WRITE (14) (PHI(I,MM   ),I=1,NJ )
      GO TO 480
  450 DO 460 I=1,NJ
  460 PHIT(I,MM   ) = PHI(I,2)
      SUM=0.
      NI=NL+1
      NZ=2*NL
      DO 470 I=NI,NZ
  470 SUM=ZSI(   2,I)+SUM
      G(   MM)=SUM/(AKSR(   2 )*ANL)
      OMN(   MM)=OMNC(2)
      ADS(   MM)=ADSR(2)
      AMS(   MM)=AMSR(2)
      AKS(   MM)=AKSR(2)
      WRITE (14) (PHIT(I,MM   ),I=1,NJ )
  480 WRITE (3,540 ) MM,OMN(MM),AMS(MM),AKS(MM),ADS(MM)
      WRITE (3,490)   ( OMN(I),I=1,NPHI)
      WRITE (3,500 )   ( AKS(I),I=1,NPHI)
      WRITE (3,510 )  ( AMS(I),I=1,NPHI)
  490 FORMAT (/////T10,'CALCULATED NATURAL FREQUENCIES, CYCLES/SEC'/
    A (1P10E13.4))
  500 FORMAT (/////T10,'CALCULATED GENERALIZED STIFFNESS'/(1P10E13.2))
  510 FORMAT (/////T10,'CALCULATED GENERALIZED MASS'/(1P10E13.2))
      REWIND 14
      DO 520 J=1,NPHI
```

```
      OMNS(J)=OMN(J)*OMN(J)
      GSQ(J)=G(J)*G(J)
  520 READ (14) (PHI(I,J),I=1,NJ )
      NQ=NPHI
      CALL ANORM (PHI,PHIM,NJ,NQ )
      WRITE (3,530 )
  530 FORMAT ('1',T50,'NORMAL MODES'// )
      CALL MOUT2 (PHIM,NJ,NQ )
      CALL PSEUDO ( PHIM,NJ,NQ, PHIA )
C
C
C     IDENTIFICATION OF MASS,STIFFNESS AND DAMPING MATRICES
      CALL TRAN (PHIA,PHIM,NQ,NJ )
  540 FORMAT (///'   MODAL PARAMETERS   MODE',I4//'     NATURAL FREQUENC
     AY='F14.3,' HERTZ'//'     GENERALIZED MASS ='F14.3,' SLUGS'//'
     BGENERALIZED STIFF='F14.2,' LB/IN'//'     GENERALIZED DAMP ='F14.2,
     C' LB-SEC/IN'/////)
C     SM=INVERSE OF MASS
C     ST=INFLUENCE COEFFICIENT
C     SI=INVERSE OF DAMPING
      DO 560 J=1,NJ
      DO 560 K=1,NQ
      SUMI=0.
      SUMM=0.
      SUMD=0.
      DO 550 I=1,NQ
      ACON=PHIM(K,I)*PHIM(J,I)
      SUMI=ACON/AKS(I)+SUMI
      SUMM=ACON/AMS(I)+SUMM
  550 SUMD=ACON/(AKS(I)*G(I))+SUMD
      ST(K,J)=SUMI
      SM(K,J)=SUMM
  560 SI(K,J)=SUMD
      CALL INVRS (SM,NJ,ZSR )
      WRITE (3,570)
  570 FORMAT ('1',T50,'IDENTIFIED MASS MATRIX'//)
      CALL MOUT2 (ZSR ,NJ,NJ )
      WRITE (3,580 )
  580 FORMAT ('1',T50,'IDENTIFIED INFLUENCE COEFFICIENT MATRIX'//)
      CALL MOUT2 (ST,NJ,NJ )
      CALL INVRS (ST,NJ,ZSR )
      WRITE (3,590)
  590 FORMAT ('1',T50,'IDENTIFIED STIFFNESS MATRIX'//)
      CALL MOUT2 ( ZSR,NJ,NJ )
      WRITE (3,600)
  600 FORMAT ('1',T50,'IDENTIFIED DAMPING MATRIX'//)
      CALL INVRS (SI,NJ,ZSR )
      CALL MOUT2 (ZSR ,NJ,NJ )
      SUM=0.
      DO 610 I=1,NQ
      WRITE (3,620) I,G(I)
  610 SUM=SUM+G(I)
      GS=SUM/NQ
  620 FORMAT (I8,F22.4)
      WRITE (3,630) GS
```

```
630 FORMAT (//'  AVG STRUCTURAL DAMPING='F8.4)
    IF (NFF.EQ.0 ) GO TO 650
640 READ (1,150 ) (HZ(I),I=1,NFF )
    NF=NFF
    GO TO 660
650 IF (NF.EQ.0) GO TO 870
660 TORF=NROW.GT.0.AND.NROW.LE.NQ
    DO 750 L=1,NF
    CON=HZ(L)*HZ(L)
    CALL MOBPHI ( G,GSQ,CON,AMS,OMNS,YR,YI,PHIM,NQ,NJ )
670 IF(IP1) 680,680,730
680 IF(IP2.NE.0) CALL MATAMP (HZ(L),YR,YI,NQ)
    IF(IP2.NE.0) GO TO 700
    WRITE (3,690) HZ(L)
690 FORMAT ('1'T40,'REAL MOBILITY, IMAGINARY MOBILITY    FREQ ='F10.2,
   A ' HERTZ'//)
    GO TO 720
700 WRITE (3,710)  HZ(L)
710 FORMAT('1'T40,'ACCELERATION AMPLITUDE IN G''S, PHASE IN DEG.  FREQ
   A ='F10.2,' HERTZ'//)
720 CALL MOUT2 (YR,NQ,NQ)
    CALL MOUT2 (YI,NQ,NQ)
    GO TO 750
730 DO 740 I=1,NQ
    DPR(L,I)=YR(I,I)
    DPI(L,I)=YI(I,I)
    IF(.NOT.TORF) GO TO 740
    TR(L,I)=YR(NROW,I)
    TI(L,I)=YI(NROW,I)
740 CONTINUE
750 CONTINUE
    IF(IP1) 870,870,760
760 IF(IP2.NE.1) GO TO 780
    CALL AMP (HZ,DPR,DPI,NF,NQ)
    IF(TORF) CALL AMP (HZ,TR,TI,NF,NQ)
    WRITE (3,770)
770 FORMAT ('1'T40,'DRIVING POINT RESPONSE,  AMP IN G''S AND PHASE IN
   ADEGREES'//)
    GO TO 810
780 WRITE (3,790)
790 FORMAT ('1'T40,'DRIVING POINT MOBILITY,  REAL AND IMAGINARY'//)
    IF ( IA.NE.0 ) WRITE (3,800)
800 FORMAT (T40,'ACCELERATION MOBILITY'//)
810 CALL YOUT (HZ,DPR,NF,NQ,0,IA )
    WRITE (3,820)
820 FORMAT ('1'//)
    CALL YOUT (HZ,DPI,NF,NQ,IP2,IA )
    IF(.NOT.TORF) GO TO 870
    IF (IP2.NE.1) GO TO 840
    WRITE (3,830) NROW
830 FORMAT ('1'T30,'TRANSFER RESPONSE, ROW 'I5,'  AMP IN G''S AND PHAS
   AE IN DEG'//)
    GO TO 860
840 WRITE (3,850) NROW
850 FORMAT ('1'T30,'TRANSFER MOBILITY, ROW 'I5,' REAL AND IMAG'//)
```

```
      IF (IA.NE.0 ) WRITE (3,800)                                       MNP 331
  860 CALL YOUT (HZ,TR,NF,NQ,0,IA )                                     MNP 332
      WRITE (3,820)                                                     MNP 333
      CALL YOUT (HZ,TI,NF,NQ, IP2,IA )                                  MNP 334
  870 CONTINUE                                                          MNP 335
      REWIND 13                                                         MNP 336
      CALL EXIT                                                         MNP 337
      END                                                               MNP 338
```

```
      SUBROUTINE TRAN ( A,B, NR,NC )                                    TRN    1
C     B=TRANSPOSE OF MATRIX A                                           TRN    2
C     A=UNDISTURBED MATRIX                                              TRN    3
      DIMENSION A(20,21),B(20,21)                                       TRN    4
      DO 100 I=1,NR                                                    1TRN    5
      DO 100 J=1,NC                                                    2TRN    6
  100 B(J,I)=A(I,J)                                                    2TRN    7
      RETURN                                                            TRN    8
      END                                                               TRN    9
```

```
      SUBROUTINE INVRS (B,N,A)
C     A = INVERSE OF B          B UNDISTURBED
C
      DIMENSION A(20,21),D(20,21),IROW(21),ICOL(21),B(20,21)
      DO 100 I=1,N
      DO 100 J=1,N
  100 A(I,J)=B(I,J)
      M=N+1
      DO 110  I=1,N
      IROW(I)=I
  110 ICOL(I)=I
      DO 26C K=1,N
      AMAX= A(K,K)
      DO 130 I=K,N
      DO 130 J=K,N
      IF(ABS( A(I,J))-ABS(AMAX))130,120,120
  120 AMAX= A(I,J)
      IC=I
      JC=J
  130 CONTINUE
      KI=ICOL(K)
      ICOL(K)=ICOL(IC)
      ICOL(IC)=KI
      KI=IROW(K)
      IROW(K)=IROW(JC)
      IROW(JC)=KI
      IF(AMAX) 160,140,160
  140 WRITE (3,150)
  150 FORMAT(' SOLUTION OF EXISTING MATRIX NOT POSSIBLE')
      GO TO 330
  160 DO 170 J=1,N
      E=A(K,J)
      A(K,J)=A(IC,J)
  170 A(IC,J)=E
      DO 180 I=1,N
      E=A(I,K)
      A(I,K)=A(I,JC)
  180 A(I,JC)=E
      DO 210 I=1,N
      IF(I-K) 200,190,200
  190 A(I,M)=1.
      GO TO 210
  200 A(I,M)=0.
  210 CONTINUE
      PVT=A(K,K)
      DO 220 J=1,M
  220 A(K,J)=A(K,J)/PVT
      DO 250 I=1,N
      IF(I-K)230,250,230
  230 AMULT=A(I,K)
      DO 240 J=1,M
  240 A(I,J)=A(I,J)-AMULT*A(K,J)
  250 CONTINUE
      DO 26C I=1,N
  260  A(I,K)=A(I,M)
```

```
      DO 290 I=1,N                                                     1INV  56
      DO 270 L=1,N                                                     2INV  57
      IF(IROW(I)-L)270,280,270                                         2INV  58
  270 CONTINUE                                                         2INV  59
  280 DO 290 J=1,N                                                     2INV  60
  290 D(L,J)=A(I,J)                                                    2INV  61
      DO 320 J=1,N                                                     1INV  62
      DO 300 L=1,N                                                     2INV  63
      IF(ICOL(J)-L) 300,310,300                                        2INV  64
  300 CONTINUE                                                         2INV  65
  310 DO 320 I=1,N                                                     2INV  66
  320 A(I,L)=D(I,J)                                                    2INV  67
  330 RETURN                                                            INV  68
      END                                                               INV  69
```

```
      SUBROUTINE MMPY (A,B,N1,N2,N3,C)
C
C                          C = A * B
C                          A (N1 X N2)    B (N2 X N3)    C (N1 X N3)
C
      REAL A(20,21),B(20,21),C(20,21)
      DO 100 I=1,N1
      DO 100 J=1,N3
      C(I,J)=0.
      DO 100 K=1,N2
  100 C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
```

```
      SUBROUTINE MOUT2 (A,M,N)
      REAL A(20,100)
      ID=MIN0(N,10)
      WRITE (3,100) (I,I=1,ID)
  100 FORMAT (/T5,10I12)
      WRITE (3,100)
      DO 110 I=1,M
  110 WRITE (3,120) I,(A(I,J),J=1,ID)
  120 FORMAT (I5,5X,1P10E12.4)
      IF (ID-N) 130,170,170
  130 ID=MIN0(N,20)
      WRITE (3,100) (I,I=11,ID )
      WRITE (3,100)
      DO 140 I=1,M
  140 WRITE (3,120) I,(A(I,J),J=11,ID )
      IF(ID-N) 150,170,170
  150 WRITE (3,100) (I,I=21,N )
      WRITE (3,100 )
      DO 160 I=1,M
  160 WRITE (3,120 ) I,( A(I,J),J=21,N )
  170 RETURN
      END
```

```
      SUBROUTINE ANORM (PHI,PHIN,NR,NC )
      DIMENSION PHI(20,21),PHIN(20,21)
      DO 120 I=1,NC
      AMAX=PHI(1,I)
      DO 100 J=2,NR
      IF(ABS(AMAX).LE.ABS(PHI(J,I)))AMAX=PHI(J,I)
  100 CONTINUE
      DO 110 J=1,NR
  110 PHIN(J,I)=PHI(J,I)/AMAX
  120 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ERRNU (A,B,PCTR,PCTBR,PCTI,PCTBI, NJ,NP,IX)
C
C                                                   A BIAS ERROR,
C             PCTB (RATIO) ON AMPLITUDE, AND A UNIFORM RANDOM ERROR
C             HAVING A +/- MAXIMUM OF PCT (RATIO) ON AMPLITUDE.
C
C
C
C        USES RANDU
C
      DIMENSION A(20,21),B(20,21)
      IF(PCTR) 110,100,110
  100 IF(PCTBR) 110,130,110
  110 DO 120 I=1,NJ
      DO 120 J=1,NP
      CALL RANDU (IX,IY,YFL)
      IX=IY
      E=1.0+2.0*PCTR*(YFL-0.5)+PCTBR
      A(I,J)=A(I,J)*E
      CALL RANDU (IX,IY,YFL)
      IX=IY
      E=1.0+2.0*PCTI *(YFL-0.5)+PCTBI
  120 B(I,J)=B(I,J)*E
  130 RETURN
      END
```

```
      SUBROUTINE RANDU (IX,IY,YFL)
C                      THIS SUBROUTINE IS FROM  SSP VERS. II
      IY=IX*65539
      IF(IY) 100,110,110
  100 IY=IY+2147483647+1
  110 YFL=IY
      YFL=YFL*.4656613E-9
      RETURN
      END
      SUBROUTINE REDI (YR,YI,NP,NJ,KEEP,INDX ,YRT,YIT )
C
C     REDUCES DISPLACEMENT MOBILITY DATA TO MATRIX OF NJ SPECIMEN
C     COORDINATES AND FORCING FREQUENCIES  Y=NJ*NP
      DIMENSION YR(20,21 ),YI(20,21 ),KEEP(20),INDX(20 )
      DIMENSION YRT(20,100),YIT(20,100)
      DO 120 I=1,NP
      DO 120 J=1,NJ
       YR(J,I)=YRT(KEEP(J),INDX(I))
  120  YI(J,I)=YIT(KEEP(J),INDX(I))
      RETURN
      END
```

```
      SUBROUTINE YOUT (OMH,A,NINC,ND,NAMP,IA )
C
C     IF IA NOT = 0 USE ACCELERATION MOBILITY
C
      REAL OMH(100),A(100,20)
      IF ( IA ) 100,120,100
  100 CON= 6.283185*6.283185
      DO 110 I=1,NINC
      DM=OMH(I)*OMH(I)*CON
      DO 110 J=1,ND
  110 A(I,J)=-A(I,J)*CM
  120 J1=1
      ID=MINO(ND,10)
  130 IL=MINO(NINC,45)
      I1=1
  140 WRITE (3,150) (I,I=J1,ID)
  150 FORMAT (T5,'HERTZ'I6,9I12)
      WRITE (3,160)
  160 FORMAT (1X)
      IF(NAMP) 170,170,200
  170 DO 180 I=I1,IL
  180 WRITE(3,190) OMH(I),(A(I,J),J=J1,ID)
  190 FORMAT (1X,F9.3,1P10E12.4)
      GO TO 230
  200 DO 210 I=I1,IL
  210 WRITE(3,220) OMH(I),(A(I,J),J=J1,ID)
  220 FORMAT (1X,F9.3,10F12.2)
  230 IF(IL-NINC) 240,260,260
  240 WRITE (3,250)
  250 FORMAT ('1'//)
      I1=46
      IL=NINC
      GO TO 140
  260 IF(ID-ND) 270,280,280
  270 J1=11
      ID=ND
      WRITE (3,220)
      GO TO 130
  280 RETURN
      END
```

```
      SUBROUTINE AMP (OMH,A,B,NINC,NR)
C
C                    CONVERTS A + I*B IN DISPLACEMENT UNITS
C                TO AMP (IN A ) IN G'S AND PHASE (IN B ) IN DEG
C                    EACH ROW IS AT A FREQUENCY OMH(I) IN HERTZ
C
C     .01626 = 6.283185 / 386.
      DIMENSION OMH(100),A(100,20),B(100,20)
C
      DO 210 I=1,NINC
      OM=OMH(I)*0.01626
      OMR=OMH(I)*6.283185
      DO 210 J=1,NR
      R=A(I,J)
      C=B(I,J)
      A(I,J)=SQRT(R*R+C*C)*OM*OMR
      IF(R) 140,100,140
  100 IF(C) 110,120,130
  110 B(I,J)=270.
      GO TO 210
  120 B(I,J)=0
      GO TO 210
  130 B(I,J)=90.
      GO TO 210
  140 P=ATAN(ABS(C/R))*57.2958
      IF(R) 150,150,180
  150 IF(C) 160,160,170
  160 B(I,J)=180.+P
      GO TO 210
  170 B(I,J)=180.-P
      GO TO 210
  180 IF(C) 190,190,200
  190 B(I,J)=360.-P
      GO TO 210
  200 B(I,J)=P
  210 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CINV (A,B,N,C,D)
C
      DIMENSION A(20,21),B(20,21),C(20,21),D(20,21),E(20,21)
C                                C+I*D = INVERSE OF A+I*B      I=SQRT(-1)
C
C                                B ASSUMED NON SINGULAR
C
      CALL  INVRS(B,N,C)
      CALL  MMPY(C,A,N,N,N,E)
      CALL  MMPY(A,E,N,N,N,C)
      DO 100 I=1,N
      DO 100 J=1,N
  100 C(I,J)=C(I,J)+B(I,J)
      CALL  INVRS(C,N,D)
      CALL  MMPY(E,D,N,N,N,C)
      DO 110 I=1,N
      DO 110 J=1,N
  110 D(I,J)=-D(I,J)
      RETURN
      END
```

```
      SUBROUTINE MOBPHI ( G,GSQ,CON,AMS,OMNS,YR,YI,PHIM,NQ,NJ )
C     CALCULATES YR AND YI USING MODAL MOBILITY AND MODE SHAPE
      DIMENSION G(20),GSQ(20),AMS(20),YR(20,20),YI(20,20),PHIM(20,20),
     AYSR(20),YSI(20),OMNS(20)
      DO 100 I=1,NQ
      CONA=CON/OMNS(I)
      CONB=1./(CON*AMS(I)*39.478413 )
      CONC=CONA-1.
      COND=CONA*CONB/(CONC*CONC+GSQ(I))
      YSR(I)=-CONC*COND
  100 YSI(I)=-G(I)*COND
      DO 120 J=1,NJ
      DO 120 K=1,NQ
      SUMR=0.
      SUMI=0.
      DO 110 I=1,NQ
      ACON=PHIM(K,I)*PHIM(J,I)
      SUMR=YSR(I)*ACON+SUMR
  110 SUMI=YSI(I)*ACON+SUMI
      YR(K,J)=SUMR
  120 YI(K,J)=SUMI
      RETURN
      END
```

```
      SUBROUTINE PSEUDO (A,NR,NC,C)
C
C             C*= PSEUDOINVERSE OF A     A UNDISTURBED
C             A IS A RECTANGULAR MATRIX OF MAXIMAL RANK  (NR X NC)
C             NR .GT. OR .LT. NC
C
C                      -1                -1
C             C =  (A'A)  A'  OR  A'(AA')
C
C             NR,NC MAY NOT EXCEED 25
C
       REAL A(20,21),B(20,21),C(20,21)
C                                         B = A'
      DO 100 I=1,NR
      DO 100 J=1,NC
  100 B(J,I)=A(I,J)
      IF(NR-NC)120,110,130
  110 CALL INVRS (A,NR,C )
      GO TO 140
C                             NR .LE. NC
C                             C = AA'
  120 CALL MMPY (A,B,NR,NC,NR,C)
C                             A = INV OF C
      CALL INVRS (C,NR,A)
C                             C = PSEUDOINVERSE OF A (NC X NR)
      CALL MMPY (B,A,NC,NR,NR,C)
      GO TO 140
C                             NC .LT. NR
C                             C = A'A
  130 CALL MMPY (B,A,NC,NR,NC,C)
C                             A = INV OF C
      CALL INVRS (C,NC,A)
C                             C = PSEUDOINVERSE OF A  (NC X NR)
      CALL MMPY (A,B,NC,NC,NR,C)
C                             RESTORE A
  140 DO 150 I=1,NR
      DO 150 J=1,NC
  150 A(I,J)=B(J,I)
      RETURN
      END
```